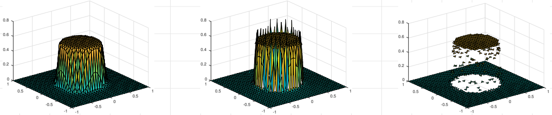
Error estimates for nonconforming and discontinuous discretizations of nonsmooth problems via convex duality



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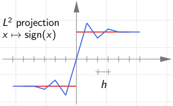
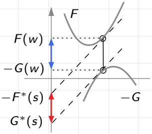
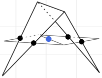
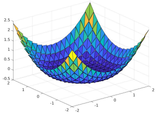
Joint work with Robert Tovey (INRIA), Zhangxian Wang, Friedrich Wassmer

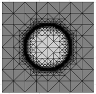






$$\left(\Pi_h \mathcal{R}T_N^0\right)^\perp = \nabla_h\left(\ker \Pi_h|_{\mathcal{S}_D^{1,cr}}\right)$$





TV Model Problem

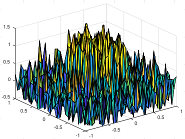
Example: Image processing via TV minimization [Rudin, Osher & Fatemi '92]

$$I(u) = |\mathrm{D}u|(\Omega) + \frac{\alpha}{2}\|u-g\|^2$$

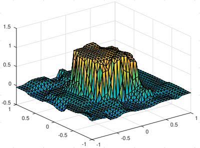
- nondifferentiable convex problem with discontinuous solutions
- formal Euler-Lagrange equations $-\operatorname{div}\frac{\nabla u}{|\nabla u|} + \alpha(u-g) = 0$

Experiment:

P1-FE
$h \sim 2^{-5}$



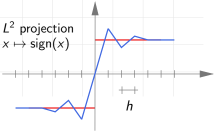
$\Longrightarrow$



Accuracy: Error estimates for L^2 error if $u \in L^\infty(\Omega) \cap BV(\Omega)$

- $\mathcal{O}(h^{1/4})$ for $P1$ FE [Wang & Lucier '11, B. '12, B., Nochetto & Salgado '14]
- $\mathcal{O}(h^{1/2})$ for anisotropic TV [B., Nochetto & Salgado '15]
- $\mathcal{O}(h^{1/2})$ for $P0/CR$ FE [Chambolle & Pock '19+, B. '20+]

Note: Rate $\mathcal{O}(h^{1/2})$ optimal for generic BV functions



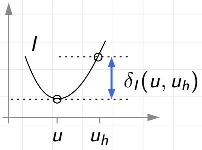
Variational problem: With convex functionals ϕ, ψ

$$I(u) = \int_\Omega \phi(\nabla u)\,\mathrm{d}x + \int_\Omega \psi(x,u)\,\mathrm{d}x$$

- formal Euler-Lagrange equations: $-\operatorname{div}\phi'(\nabla u) + \psi'(\cdot, u) = 0$

Coercivity: Optimality of u

$$\delta_I(u, u_h) \le I(u_h) - I(u)$$



Error estimate: By minimality of u_h for every v_h

$$\delta_I(u, u_h) \le I(v_h) - I(u)$$

Hence: Suitable interpolant $v_h = \mathcal{J}_h u$ leads to error estimate

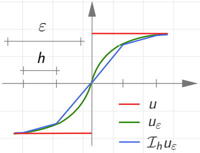
- control of inconsistent $I_h \approx I$
- optimality and regularity
- use of duality $I(u) \ge D(z)$

Model: ROF minimization in $BV(\Omega)$, discretization with $P1$-FE

$$\frac{\alpha}{2}\|u - u_h\|^2 \le I(v_h) - I(u)$$

Interpolant: With regularization u_ε set $v_h = \mathcal{I}_h u_\varepsilon$

$$\|u - \mathcal{I}_h u_\varepsilon\|_{L^1} \le c(h^2\varepsilon^{-1} + \varepsilon)|Du|(\Omega),$$
$$\|\nabla \mathcal{I}_h u_\varepsilon\|_{L^1} \le (1 + ch\varepsilon^{-1} + c\varepsilon)|Du|(\Omega)$$



Continuity: With binomial formula and $\varepsilon = h^{1/2}$

$$\frac{\alpha}{2}\|u - u_h\|^2 \le |Dv_h|(\Omega) - |Du|(\Omega) + \frac{\alpha}{2}\int_\Omega (v_h - u)(v_h + u - 2g)\,\mathrm{d}x$$
$$\le ch\varepsilon^{-1} + c(h^2\varepsilon^{-1} + \varepsilon) = \mathcal{O}(h^{1/2})$$

Optimality via TVD: If $\widetilde{\mathcal{I}}_h : W^{1,1}(\Omega) \to \mathcal{S}^1(\mathcal{T}_h)$ with $\|\nabla \widetilde{\mathcal{I}}_h v\|_{L^1} \le \|\nabla v\|_{L^1}$ then

$$\frac{\alpha}{2}\|u - u_h\|^2 = \mathcal{O}(h)$$

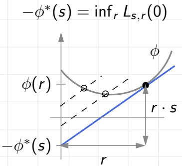
OK: Anisotropic ℓ^1 version of BV and structured grids [B., Nochetto & Salgado '15]

Convexity and Nonstandard FE

Conjugate: For convex $\phi : \mathbb{R}^\ell \to \mathbb{R}$ define *convex conjugate*

$$\phi^*(s) = \sup_{r \in \mathbb{R}^\ell} r \cdot s - \phi(r)$$

i.e., Young's ineq $r \cdot s \le \phi(r) + \phi^*(s)$, equality for $s = \phi'(r)$

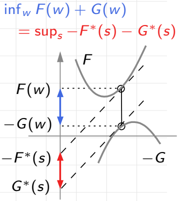


Duality: Use $\phi^{**} = \phi$, integrate by parts, exchange extrema

$$\inf_u I(u) = \inf_u \int_\Omega \overbrace{\sup_z \nabla u \cdot z - \phi^*(z)}^{=\phi(\nabla u)} \,\mathrm{d}x + \int_\Omega \psi(x,u) \,\mathrm{d}x$$

$$\ge \sup_z \inf_u - \int_\Omega \phi^*(z) \,\mathrm{d}x - \int_\Omega u \operatorname{div} z - \psi(x,u) \,\mathrm{d}x$$

$$= \sup_z - \int_\Omega \phi^*(z) \,\mathrm{d}x - \int_\Omega \psi^*(x, \operatorname{div} z) \,\mathrm{d}x = \sup_z D(z)$$



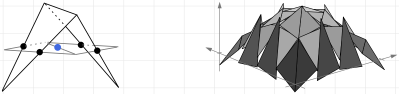
Strong duality: If saddle (u, z) exists then equality $I(u) = D(z)$ holds

- important implicit information in dual solution $z = \phi'(\nabla u)$

Motivation: Continuous $P1$ vector fields too stiff for divergence, e.g. in Stokes,

$$\operatorname{div} : \mathcal{S}^1(\mathcal{T}_h)^d \to \mathcal{L}^0(\mathcal{T}_h) \quad \text{NOT surjective}$$

CR-FEM: $\mathcal{S}^{1,cr}(\mathcal{T}_h)$ continuous (only) at side midpoints [Crouzeix–Raviart '73]



Quasi-interpolant: Averages over element sides

$$\mathcal{J}_h : H^1(\Omega) \to \mathcal{S}^{1,cr}(\mathcal{T}_h), \quad \mathcal{J}_h v(x_S) = |S|^{-1} \int_S v \,\mathrm{d}s$$

Projection property: Elementwise gradient given by projection onto constants

$$\nabla_h \mathcal{J}_h v = \Pi_h \nabla v$$

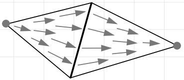
Jensen's inequality: For convex function ϕ

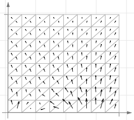
$$\int_\Omega \phi(\nabla_h \mathcal{J}_h v) \,\mathrm{d}x \le \int_\Omega \phi(\nabla v) \,\mathrm{d}x$$

Motivation: Divergence operator on L^2 vector fields (instead H^1)

RT-FEM: $\mathcal{RT}^0(\mathcal{T}_h)$ affine vector fields with weak divergence [Raviart–Thomas '77]

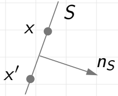
$$q_h|_T(x) = ax + b$$
$$a \in \mathbb{R},\ b \in \mathbb{R}^d$$





Idea: Constant normal components on hyperplanes

$$q_h(x) \cdot n_S = c \quad \forall x \in S$$



Quasi-interpolant: Averages of normal components over sides

$$\mathcal{L}_h : H(\mathrm{div};\Omega) \to \mathcal{RT}^0(\mathcal{T}_h), \quad \mathcal{L}_h q(x_S) \cdot n_S = |S|^{-1} \int_S q \cdot n_S \,\mathrm{d}s$$

Projection property: Surjectivity of $\mathrm{div} : \mathcal{RT}^0(\mathcal{T}_h) \to \mathcal{L}^0(\mathcal{T}_h)$ via

$$\mathrm{div}\, \mathcal{L}_h q = \Pi_h \,\mathrm{div}\, q$$

Properties of CR and RT: On element sides $S \in \mathcal{S}_h$ with normal n_S

- jumps $[\![v_h]\!]$ of CR functions over sides have vanishing mean
- components $q_h \cdot n_S$ for RT fields constant and continuous

Integration by parts: For $v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ and $q_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$

$$\int_\Omega v_h \operatorname{div} q_h \,\mathrm{d}x = -\sum_{T \in \mathcal{T}_h} \int_T \nabla v_h \cdot q_h \,\mathrm{d}x + \sum_{S \in \mathcal{S}_h} \underbrace{\int_S [\![v_h]\!] q_h \cdot n_S \,\mathrm{d}s}_{=0}$$
$$= -\int_\Omega \nabla_h v_h \cdot q_h \,\mathrm{d}x$$

Projection onto constants: L^2 projection $\Pi_h : L^2(\Omega;\mathbb{R}^\ell) \to \mathcal{L}^0(\mathcal{T}_h)^\ell$

Duality connection: Within $\mathcal{L}^0(\mathcal{T}_h)^\ell$

$$\int_\Omega \Pi_h v_h \operatorname{div} q_h \,\mathrm{d}x = -\int_\Omega \nabla v_h \cdot \Pi_h q_h \,\mathrm{d}x$$

- projection onto constants important to apply Fenchel's inequality

Imitate arguments: For $u_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ and $z_h \in \mathcal{R}T^0_N(\mathcal{T}_h)$

$$\begin{aligned} I_h(u_h) &= \int_\Omega \phi(\nabla_h u_h) + \psi_h(\Pi_h u_h)\,\mathrm{d}x \\ &\geq \int_\Omega -\phi^*(\Pi_h z_h) + \Pi_h z_h \cdot \nabla_h u_h + \psi_h(\Pi_h u_h)\,\mathrm{d}x \\ &= \int_\Omega -\phi^*(\Pi_h z_h) - \operatorname{div} z_h\, \Pi_h u_h + \psi_h(\Pi_h u_h)\,\mathrm{d}x \\ &\geq \int_\Omega -\phi^*(\Pi_h z_h) - \psi_h^*(\operatorname{div} z_h)\,\mathrm{d}x \\ &= D_h(z_h) \end{aligned}$$

Weak duality: Discrete functionals I_h and D_h satisfy $I_h(u_h) \geq D_h(z_h)$

Strong duality: If ϕ, $\psi_h \in C^1$ and u_h minimal for I_h then

$$z_h = D\phi(\nabla_h u_h) + D\psi_h(\cdot, \Pi_h z_h) d^{-1}(x - x_T) \quad \Longrightarrow \quad I_h(u_h) = D_h(z_h)$$

Related: $z_h = D\phi(\nabla_h u_h) + d^{-1}(x - x_T)\Pi_h f$ [Marini '85, Carstensen & Liu '15]

Nonlinear Dirichlet problem: With $f_h = \Pi_h f$ consider

$$I_h(u_h) = \int_\Omega \phi(\nabla_h u_h)\,\mathrm{d}x - \int_\Omega f_h\,\Pi_h u_h\,\mathrm{d}x,$$

$$D_h(z_h) = -\int_\Omega \phi^*(\Pi_h z_h)\,\mathrm{d}x - I_{\{-f_h\}}(\operatorname{div} z_h)$$

A priori analysis: Control discrete error $\delta_{I_h}(u_h, \mathcal{J}_h u)$ via

$$\delta_{I_h} \le I_h(\mathcal{J}_h u) - I_h(u_h) \le I_h(\mathcal{J}_h u) - D_h(\mathcal{L}_h z)$$

- discrete coercivity & duality

$$\le \int_\Omega \phi(\nabla u) - \mathcal{L}_h z_h \cdot \nabla_h \mathcal{J}_h u + \phi^*(\Pi_h \mathcal{L}_h z)\,\mathrm{d}x$$

- Jensen's inequality, $\operatorname{div} \mathcal{L}_h z = -f_h$

$$\le \int_\Omega -\phi^*(z) + (z - \Pi_h \mathcal{L}_h z)\cdot \nabla u + \phi^*(\Pi_h \mathcal{L}_h z)\,\mathrm{d}x$$

- strong duality $I(u) = D(z)$, $\operatorname{div} z = -f$, $\nabla_h \mathcal{J}_h u = \Pi_h \nabla u$

$$\le \int_\Omega \big(D\phi^*(z) - D\phi^*(\Pi_h \mathcal{L}_h z)\big)\cdot (z - \Pi_h \mathcal{L}_h z)\,\mathrm{d}x$$

- convexity, $\nabla u = D\phi^*(z)$

$$= \mathcal{O}(h^2)$$

- interpolation estimates, if, e.g., ϕ^* Lipschitz

Generalization: Convex variational problems on $W^{1,p}_D(\Omega)$

Theorem ([B. '20+]). If $z \in W^q_N(\mathrm{div};\Omega) \cap W^{1,1}(\Omega;\mathbb{R}^d)$

$$
\begin{aligned}
\delta_{I_h}(u_h, \mathcal{J}_h u) \le \int_\Omega & \big(D\phi^*(z) - D\phi^*(\Pi_h \mathcal{L}_h z)\big) \cdot (z - \Pi_h \mathcal{L}_h z) \,\mathrm{d}x \\
&+ \int_\Omega \big(D\psi(u) - D\psi_h(\Pi_h \mathcal{J}_h u)\big) \cdot (u - \Pi_h \mathcal{J}_h u) \,\mathrm{d}x \\
&+ \int_\Omega \psi_h(u) - \psi(u) \,\mathrm{d}x + \int_\Omega \psi_h^*(\Pi_h \operatorname{div} z) - \psi^*(\operatorname{div} z) \,\mathrm{d}x.
\end{aligned}
$$

Interpretation: CR discretization error controlled by RT interpolation error

- inconsistency $\nabla_h \approx \nabla$ and $I_h \approx I$ estimated via duality
- low order terms reduce to $(\gamma/2)\|u - \Pi_h \mathcal{J}_h u\|^2$ for quadratic ψ
- avoids use of Strang lemmas; guarantee feasibility of interpolants
- use of quadrature important and computationally efficient

Orthogonality relation: Within $(\mathcal{L}^0)^d \subset L^2(\Omega;\mathbb{R}^d)$ [B. & Wang '20+]

$$(\Pi_h \mathcal{R}T_N^0)^\perp = \nabla_h\big(\ker \Pi_h|_{\mathcal{S}_D^{1,cr}}\big) \quad \Longleftrightarrow: \quad A^\perp = B$$

Proof: (i) If $\Pi_h v_h = 0$ then for all $y_h \in \mathcal{R}T_N^0$

$$(\nabla_h v_h, \Pi_h y_h) = -(\Pi_h v_h, \operatorname{div} y_h) = 0 \implies A \subset B^\perp$$

(ii) Let $y_h \in B^\perp$. Choose $r_h \in Z_h = (\ker \Pi_h|_{\mathcal{S}_D^{1,cr}})^\perp \subset \mathcal{S}_D^{1,cr}$ with

$$(\Pi_h r_h, \Pi_h v_h) = (y_h, \nabla_h v_h) \quad \forall v_h \in Z_h.$$

True for *all* $v_h \in \mathcal{S}_D^{1,cr}$. Let $z_h \in \mathcal{R}T_N^0$ with $-\operatorname{div} z_h = \Pi_h r_h$

$$(y_h - z_h, \nabla_h v_h) = (\Pi_h r_h, \Pi_h v_h) + (\operatorname{div} z_h, v_h) = 0 \quad \forall v_h \in \mathcal{S}_D^{1,cr}$$

Define $\widetilde{y}_h|_T = y_h|_T + \operatorname{div} z_h|_T (x - x_T)/d$ so that

$$(\widetilde{y}_h - z_h, \nabla_h v_h) = (y_h - z_h, \nabla_h v_h) = 0 \quad \forall v_h \in \mathcal{S}_D^{1,cr}$$

Implies $\widetilde{y}_h - z_h \in \mathcal{R}T_N^0$. Since $\Pi_h \widetilde{y}_h = y_h$ deduce $y_h \in A$ and $B^\perp \subset A$. □

Remarks: Identity uses inf-sup condition and has consequences

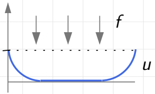
- strong discrete duality relation
- dual variant $\operatorname{div} \ker \Pi_h|_{\mathcal{R}T_N^0} = (\Pi_h \mathcal{S}_D^{1,cr})^\perp \implies \Pi_h \mathcal{S}_D^{1,cr} = \mathcal{L}^0$ if $\Gamma_D \neq \partial\Omega$
- alternative proof uses discrete Poincaré lemma [Chambolle & Pock '19+]

Nonsmooth Examples

Functionals: For $f \in L^2(\Omega)$ impose $u \geq 0$ via indicator functionals I_+

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \int_\Omega f u \,\mathrm{d}x + I_+(u),$$

$$D(z) = -\frac{1}{2}\int_\Omega |z|^2 \,\mathrm{d}x - I_-(\operatorname{div} z + f)$$



Low order terms: Pointwise convex conjugation

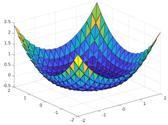
$$\psi(x,s) = -f(x)s + I_+(s) \quad \Longrightarrow \quad \psi^*(x,t) = I_-(t + f(x))$$

Complementarity: $z = \nabla u$ and $\operatorname{div} z + f = 0$ if $u > 0$

Discretization: Use $f_h = \Pi_h f$ and impose constraint at midpoints of elements

$$I_h(u_h) = \frac{1}{2}\int_\Omega |\nabla_h u_h|^2 \,\mathrm{d}x - \int_\Omega f_h u_h \,\mathrm{d}x + I_+(\Pi_h u_h),$$

$$D_h(z_h) = -\frac{1}{2}\int_\Omega |\Pi_h z_h|^2 \,\mathrm{d}x - I_-(\operatorname{div} z_h + f_h)$$



Proposition ([B. '20+]). If $z = \nabla u \in H^1(\Omega;\mathbb{R}^d)$ then

$$\|\nabla_h(u_h - u)\| \le ch\big(\|D^2 u\| + \|f + \operatorname{div} z\|\big).$$

Proof: Since $\mathcal{J}_h u$ and $\mathcal{L}_h z$ admissible with $\mu = f + \operatorname{div} z$

$$\delta_h^2 \le \int_\Omega (f + \operatorname{div} z)(u - \Pi_h \mathcal{J}_h u)\,\mathrm{d}x + \frac{1}{2}\int_\Omega |z - \Pi_h \mathcal{L}_h z|^2\,\mathrm{d}x$$
$$= \int_\Omega \mu(\mathcal{J}_h u - \Pi_h \mathcal{J}_h u)\,\mathrm{d}x + \int_\Omega \mu(u - \mathcal{J}_h u)\,\mathrm{d}x + \frac{1}{2}\int_\Omega |z - \Pi_h \mathcal{L}_h z|^2\,\mathrm{d}x.$$

For element contact set $\mathcal{C}_T = \{x \in T : u(x) = 0\}$

$$\mu|_{T\setminus\mathcal{C}_T} = 0, \quad \nabla u|_{\mathcal{C}_T} = 0.$$

With $\mathcal{J}_h u(x) = \mathcal{J}_h u(x_T) + \nabla_h \mathcal{J}_h u|_T \cdot (x - x_T)$ for critical term A

$$A|_T = \int_{\mathcal{C}_T} \mu\,\nabla_h(\mathcal{J}_h u - u)\cdot(x - x_T)\,\mathrm{d}x \le h_T \|\mu\|_{L^2(T)} \|\nabla_h(\mathcal{J}_h u - u)\|_{L^2(T)}. \qquad \square$$

- note constant-free intermediate estimate

Primal problem: For noisy image $g \in L^\infty(\Omega)$ find u via minimizing

$$I(u) = |\mathrm{D}u|(\Omega) + \frac{\alpha}{2}\|u - g\|^2$$

Total variation: Extends $\|\nabla u\|_{L^1(\Omega)}$ via operator norm of Du on $C(\overline{\Omega})$

$$|\mathrm{D}u|(\Omega) = \sup\Big\{ -\int_\Omega u \operatorname{div} z \,\mathrm{d}x : z \in C_c^\infty(\Omega;\mathbb{R}^d),\ |z|_{\ell^2} \le 1 \text{ in } \Omega \Big\}$$

Dual problem: By exchanging extrema $u = g + \alpha^{-1} \operatorname{div} z$

$$D(z) = -\frac{1}{2\alpha}\|\operatorname{div} z + \alpha g\|^2 + \frac{\alpha}{2}\|g\|^2 - I_{K_1(0)}(z)$$

Optimality: $z \in \partial|\nabla u|$ and strong duality $I(u) = D(z)$ [Hintermüller & Kunisch '04]

Discretization: With $g_h = \Pi_h g$ consider inconsistent I_h and D_h

$$I_h(u_h) = \int_\Omega |\nabla_h u_h| \,\mathrm{d}x + \frac{1}{2}\|\Pi_h u_h - g_h\|^2,$$

$$D_h(z_h) = -\frac{1}{2\alpha}\|\operatorname{div} z_h + \alpha g_h\|^2 + \frac{\alpha}{2}\|g_h\|^2 - I_{K_1(0)}(\Pi_h z_h)$$

Proposition ([Chambolle & Pock '19+, B. '20+]). If $z \in W^{1,\infty}(\Omega;\mathbb{R}^d)$ then

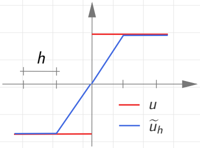
$$\|u - \Pi_h u_h\| \le c h^{1/2} \big(\|u\|_{L^\infty} |\mathrm{D}u|(\Omega) + \|g\| \|\nabla z\|_{L^\infty} \|\operatorname{div} z\|\big)^{1/2}.$$

Proof: For admissible $\widetilde{u}_h$ and $\widetilde{z}_h$

$$\frac{1}{2}\|\Pi_h(u_h - \widetilde{u}_h)\|^2 \le I_h(\widetilde{u}_h) - I_h(u_h) \le I_h(\widetilde{u}_h) - D_h(\widetilde{z}_h)$$

For $\widetilde{u}_h = \lim_{\varepsilon\to 0} \mathcal{J}_h u_\varepsilon$ have TVD and approximation

$$\|\nabla_h \widetilde{u}_h\|_{L^1} \le |\mathrm{D}u|(\Omega), \quad \|\widetilde{u}_h - u\|_{L^1} \le ch|\mathrm{D}u|(\Omega),$$



and hence

$$I_h(\widetilde{u}_h) \le I(u) + ch - \frac{1}{2}\|g - g_h\|^2.$$

For $\widetilde{z}_h = \gamma_h^{-1} \mathcal{L}_h z$ with $\gamma_h = \|\mathcal{L}_h z\|_{L^\infty}$ have

$$D_h(\widetilde{z}_h) \ge D(z) - ch - \frac{1}{2}\|g - g_h\|^2.$$

Strong duality $I(u) = D(z)$ gives estimate. □

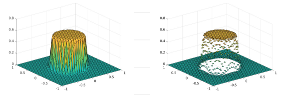
- note canceling low order contributions; no regularity conditions on g

Setup: For Dirichlet BC, $g = \chi_{B_r(0)}$ have

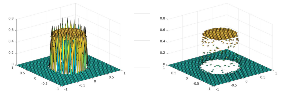
$$u = \max\{0, 1 - d/(\alpha r)\}\chi_{B_r(0)}, \quad z \in W^{1,\infty}(\Omega; \mathbb{R}^d)$$

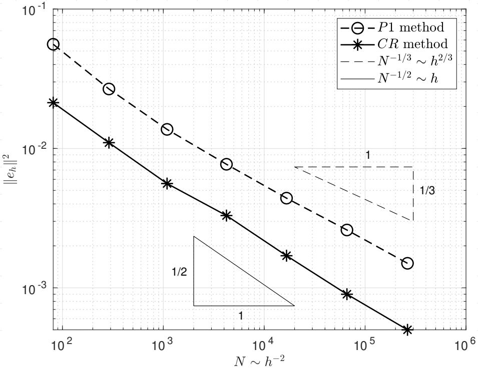
- choose $r = 1/2$, $d = 2$, $\Omega = (-1, 1)^2$, and $\alpha = 10$
- iterative solution via semi-implicit gradient descent [B., Diening & Nochetto '18]

$P1$ solution u_h^{p1} and $\Pi_h u_h^{p1}$



CR solution u_h^{cr} and $\Pi_h u_h^{cr}$





Single disc phantom **[Chambolle et al. '10]:** For $B = B_1(0) \subset\subset \Omega$ and $g = \chi_B$, Dirichlet BC, ROF minimizer $u = c_\alpha g$ with Lipschitz continuous dual

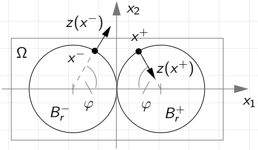
$$z(x) = -c'_\alpha \begin{cases} x, & |x| \le 1 \\ x/|x|^2, & |x| \ge 1 \end{cases}$$

Optimality: Dual solutions satisfy $z \in \partial|Du|$ and $\operatorname{div} z = -\alpha(u-g)$

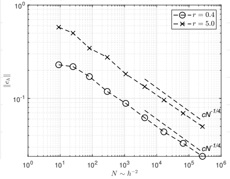
Touching discs **[B., Tovey & Wassmer '21+]:**
For symmetric Ω, $g = \chi_{B^+} - \chi_{B^-}$, Dirichlet BC, have $u = c'_{1,\alpha} g$ and $z = \mp\nu$ on $\partial B^\pm$ and

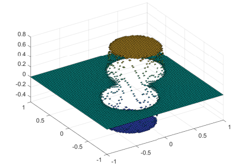
$$|x^+ - x^-| \approx \varphi^2, \quad |z(x^+) - z(x^-)| \approx 2\varphi$$

$\Longrightarrow$ z not Lipschitz continuous



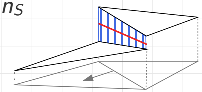
No reduction of CR convergence rate:





Generalization to dG

Jumps and averages: On inner sides $S \in \mathcal{S}_h$ with normal n_S



$$[\![v_h]\!](x) = \lim_{\varepsilon \to 0} \big(v_h(x - \varepsilon n_S) - v_h(x + \varepsilon n_S)\big),$$

$$\{v_h\}(x) = \lim_{\varepsilon \to 0} \frac{1}{2}\big(v_h(x - \varepsilon n_S) + v_h(x + \varepsilon n_S)\big)$$

- midpoint evaluation $[\![v_h]\!]_h|_S = [\![v_h]\!](x_S)$ and $\{v_h\}_h|_S = \{v_h\}(x_S)$

Broken CR and RT spaces: Omit all continuity requirements

$$\mathcal{S}^{1,dg}(\mathcal{T}_h) = \mathcal{L}^1(\mathcal{T}_h), \quad \mathcal{R}T^{0,dg}(\mathcal{T}_h) = \mathcal{L}^0(\mathcal{T}_h)^d + (\mathrm{id} - x_{\mathcal{T}})\mathcal{L}^0(\mathcal{T}_h)$$

Integration by parts: With $[\![v_h y_h]\!] = [\![v_h]\!]\{y_h\} + \{v_h\}[\![y_h]\!]$

$$\int_\Omega v_h \operatorname{div}_h y_h \,\mathrm{d}x + \int_\Omega \nabla_h v_h \cdot y_h \,\mathrm{d}x = \int_{\mathcal{S}_h \setminus \Gamma_N} [\![v_h]\!]_h \{y_h \cdot n_S\} \,\mathrm{d}s + \int_{\mathcal{S}_h \setminus \Gamma_D} \{v_h\}_h [\![y_h \cdot n_S]\!] \,\mathrm{d}s$$

- reveals duality of jumps and averages

Stabilizing and regularizing terms: For $u_h \in \mathcal{S}^{1,dg}(\mathcal{T}_h)$ and $z_h \in \mathcal{R}T^{0,dg}(\mathcal{T}_h)$

$$J_h(u_h) = \frac{1}{r}\|\alpha_{\mathcal{S}}^{-1}[\![u_h]\!]_h\|_{L^r(\mathcal{S}_h\setminus\Gamma_N)}^r + \frac{1}{s}\|\beta_{\mathcal{S}}\{u_h\}_h\|_{L^s(\mathcal{S}_h\setminus\Gamma_D)}^s,$$

$$K_h(z_h) = \frac{1}{r'}\|\alpha_{\mathcal{S}}\{z_h\cdot n_{\mathcal{S}}\}\|_{L^{r'}(\mathcal{S}_h\setminus\Gamma_N)}^{r'} + \frac{1}{s'}\|\beta_{\mathcal{S}}^{-1}[\![z_h\cdot n_{\mathcal{S}}]\!]\|_{L^{s'}(\mathcal{S}_h\setminus\Gamma_D)}^{s'}$$

Proposition ([B. '20+]). $I_h(u_h) \geq D_h(z_h)$ for primal and dual dG functionals

$$I_h(u_h) = \int_\Omega \phi(\nabla_h u_h) + \psi_h(x, \Pi_h u_h)\,\mathrm{d}x + J_h(u_h),$$

$$D_h(z_h) = -\int_\Omega \phi^*(\Pi_h z_h) + \psi_h^*(x, \operatorname{div}_h z_h)\,\mathrm{d}x - K_h(z_h).$$

If $r = s = 2$ and $\beta_{\mathcal{S}} = 0$ and $\phi, \psi \in C^1$ then strong duality applies.

- use of quadrature in jumps and averages essential

Nonlinear Dirichlet problems: Consider low order term $\psi(x,s) = -f(x)s$

Proposition ([B. '20+]). If $z \in W^{1,1}(\Omega;\mathbb{R}^d)$

$$\delta_{I_h}(u_h, \mathcal{J}_h u) \le \int_\Omega \big(D\phi^*(z) - D\phi^*(\Pi_h \mathcal{L}_h z)\big) \cdot (z - \Pi_h \mathcal{L}_h z)\,\mathrm{d}x + J_h(\mathcal{J}_h u) + K_h(\mathcal{L}_h z),$$

where

$$J_h(\mathcal{J}_h u) \le c_{\mathcal{T}} \|h_{\mathcal{S}}^{-1} \beta_{\mathcal{S}}^s\|_{L^\infty(\mathcal{S})} \|\mathcal{J}_h u\|_{L^s(\Omega)}^s,$$

$$K_h(\mathcal{L}_h z) \le c_{\mathcal{T}} \|h_{\mathcal{S}}^{-1} \alpha_{\mathcal{S}}^{r'}\|_{L^\infty(\mathcal{S})} \|\mathcal{L}_h z\|_{L^{r'}(\Omega)}^{r'}.$$

Examples: Strength of jump terms

- Poisson: $r = 2$ and $\alpha_S = h^{3/2}$ (sharp)
- TV: $r = 1$ and $\alpha_S \le 1$ or $r = 2$ and $\alpha_S = h^p$, $p \ge 2$

Consistency: Discretize then optimize

- note disappearing jump terms for CR and RT comparison functions
- lack of consistency terms for elliptic problems requires overpenalization

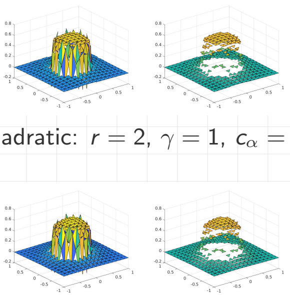
Regularized functional: With $|a|_\varepsilon = (|a|^2 + \varepsilon^2)^{1/2}$

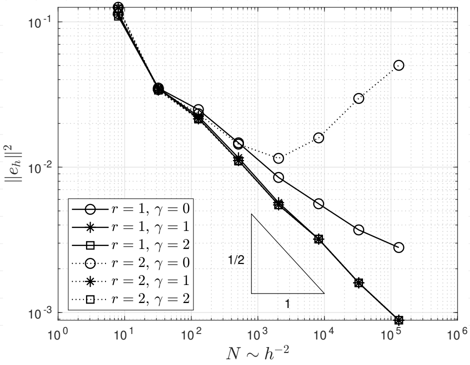
$$I_{h,\varepsilon}(u_h) = \int_\Omega |\nabla_h u_h|_\varepsilon \,\mathrm{d}x + \frac{\alpha}{2}\|\Pi_h u_h - g_h\|^2 + \frac{c_\alpha^{-r}}{r}\int_{\mathcal{S}} h_{\mathcal{S}}^{-\gamma r} |[\![u_h]\!]|_\varepsilon^r \,\mathrm{d}s$$

- uniform estimate $0 \le |a|_\varepsilon - |a| \le \varepsilon$ justifies $\varepsilon = h$

Linear: $r = 1,\ \gamma = 1,\ c_\alpha = 1$

Quadratic: $r = 2,\ \gamma = 1,\ c_\alpha = 1$





Mesh Grading and Adaptivity

L^2 Best approximation: For $u = \mathrm{sign} + v$, $v \in H^1(a,b)$

$$c_1 h_0^{1/2} \le \inf_{u_h \in \mathcal{S}^1(\mathcal{T}_h)} \|u - u_h\| \le c_2\big(h_0^{1/2} + h_{\max}\|v'\|\big)$$

Idea: Use β-graded mesh with $h_{\min} \sim h_{\max}^{\beta}$

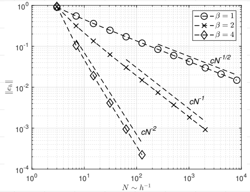
1D ROF model: If $u \in W^{2,1}((a,b)\setminus J_u)$

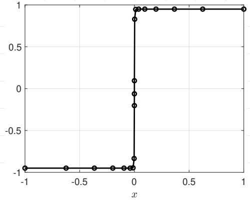
$$\frac{\alpha}{2}\|u - u_h\|^2 \le I(\mathcal{I}_h u) - I(u)$$

$$= \underbrace{\|(\mathcal{I}_h u)'\|_{L^1(a,b)} - |Du|(a,b)}_{\le 0 \text{ (TVD property of } \mathcal{I}_h)}$$

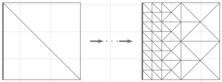
$$+ \frac{\alpha}{2} \underbrace{\int_a^b (\mathcal{I}_h u - g)^2 - (u-g)^2 \,\mathrm{d}x}_{\le 4\|u - \mathcal{I}_h u\|_{L^1}\|g\|_{L^\infty}}$$

$$\le c\alpha\big(h_{\min}|Du|(a,b) + h_{\max}^2\|u''\|_{L^1((a,b)\setminus J_u)}\big)\|g\|_{L^\infty}$$





Graded triangulations: Shape regularity and conformity limit grading strength in $\mathbb{R}^d$, $d > 1$



CR discretization: Use discrete TVD property and discrete duality as above

$$\frac{\alpha}{2}\|\Pi_h(\widetilde{u}_h - u_h)\|^2 \le 2\alpha\|\Pi_h\widetilde{u}_h - u\|_{L^1(\Omega)}\|g\|_{L^\infty(\Omega)} + (1-\gamma_h^{-1})\|g\|\|\operatorname{div} z\|$$

Difficulty: Local error $\mathcal{O}(h_T)$ even in regular regions, $\gamma_h \ge \max_{T\in\mathcal{T}_h}|J_{RT}z(x_T)|$

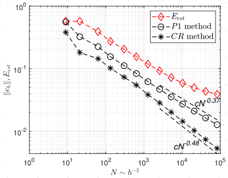
Proposition ([B., Tovey & Wassmer '21+]). If ROF sol. $u \in BV(\Omega)$ p/w constant, dual $z \in W^{1,\infty}(\Omega;\mathbb{R}^d)$ s.t.

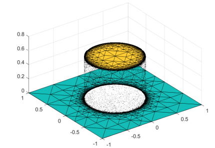
$$|z(x)| \le 1 - \ell d_{J_u}(x)$$

and triangulations quadratically graded

$$h_T \le c \begin{cases} h d_{J_u}(T)^{1/2} & \text{if } d_{J_u}(T) \ge h^2, \\ h^2 & \text{otherwise} \end{cases}$$

then $\|\Pi_h(u-\widetilde{u}_h)\| \le h|\log h|\, M(\alpha,u,z,g)$.





Continuous coercivity: For conforming approximation $u_h^c \in W^{1,p}(\Omega)$

$$\begin{aligned}\delta_I(u,u_h^c) \le I(u_h^c) - I(u) \le I(u_h^c) - D(z_h) \\ = \int_\Omega \phi(\nabla u_h^c) + \psi(u_h^c) + \phi^*(z_h) + \psi^*(\operatorname{div} z_h)\,\mathrm{d}x\end{aligned}$$

Poisson problem: Assume $f_h = f$ and use $\operatorname{div} z_h = -f$

$$\frac{1}{2}\|\nabla(u-u_h^c)\|^2 \le \frac{1}{2}\int_\Omega |\nabla u_h^c|^2 + |z_h|^2 + \operatorname{div} z\, u_h^c \,\mathrm{d}x = \frac{1}{2}\|\nabla u_h^c - z_h\|^2$$

TV minimization: Direct calculations and integration by parts for $|z_h| \le 1$

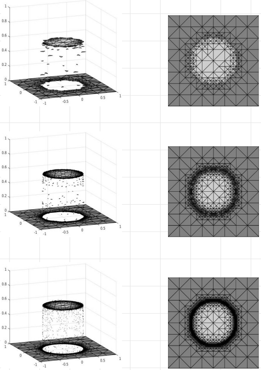
$$\begin{aligned}\frac{\alpha}{2}\|u-u_h^c\|^2 &\le \int_\Omega |\nabla u_h^c| + \frac{\alpha}{2}(u_h^c - g)^2 + \frac{1}{2\alpha}(\operatorname{div} z + \alpha g)^2 - \frac{\alpha}{2} g^2 \,\mathrm{d}x \\ &= \int_\Omega \big(|\nabla u_h^c| - \nabla u_h^c \cdot z_h\big) + \frac{1}{2\alpha}\big(\operatorname{div} z_h + \alpha(u_h^c - g)\big)^2 \,\mathrm{d}x\end{aligned}$$

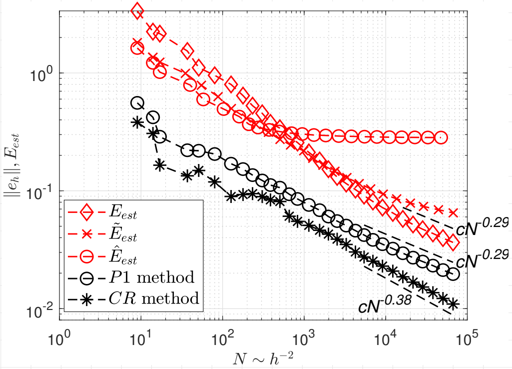
Interpretation: Residual type estimators, u_h^c and z_h only admissible

- flux equilibration, hypercircle method: [Repin '00, Braess '07, B. '15]
- nearly optimal z_h via postprocessing primal CR approximation u_h

ROF estimator: Use localized error bound to refine mesh [B. & Milicevic '19]

$$e_h^2 \le \eta_h^2 = \sum_{T \in \mathcal{T}_h} \int_T \left(|\nabla u_h^c| - \nabla u_h^c \cdot z_h\right) + \frac{1}{2\alpha}\left(\operatorname{div} z_h + \alpha(u_h^c - g)\right)^2 \,\mathrm{d}x$$





- use of $H(\operatorname{div};\Omega)$-conforming dual variable avoids oscillations
- open: construction of admissible, nearly optimal $z^{(h)}$

Summary

- Optimal error estimates via convex duality
- Nonsmooth problems, only first order systems
- Dual problem provides right regularity condition
- A posteriori error estimates via primal-dual gap
- Solve dual problem via generalized Marini formula

- Future aspects/open problems:
  - general regularity results, e.g., dual form TV
  - higher order nonstandard finite elements
  - optimal iterative solution

- More information

`http://aam.uni-freiburg.de/bartels`

Thank you !