Detailed Simulation of Viral Propagation and Mitigation in the Built Environment

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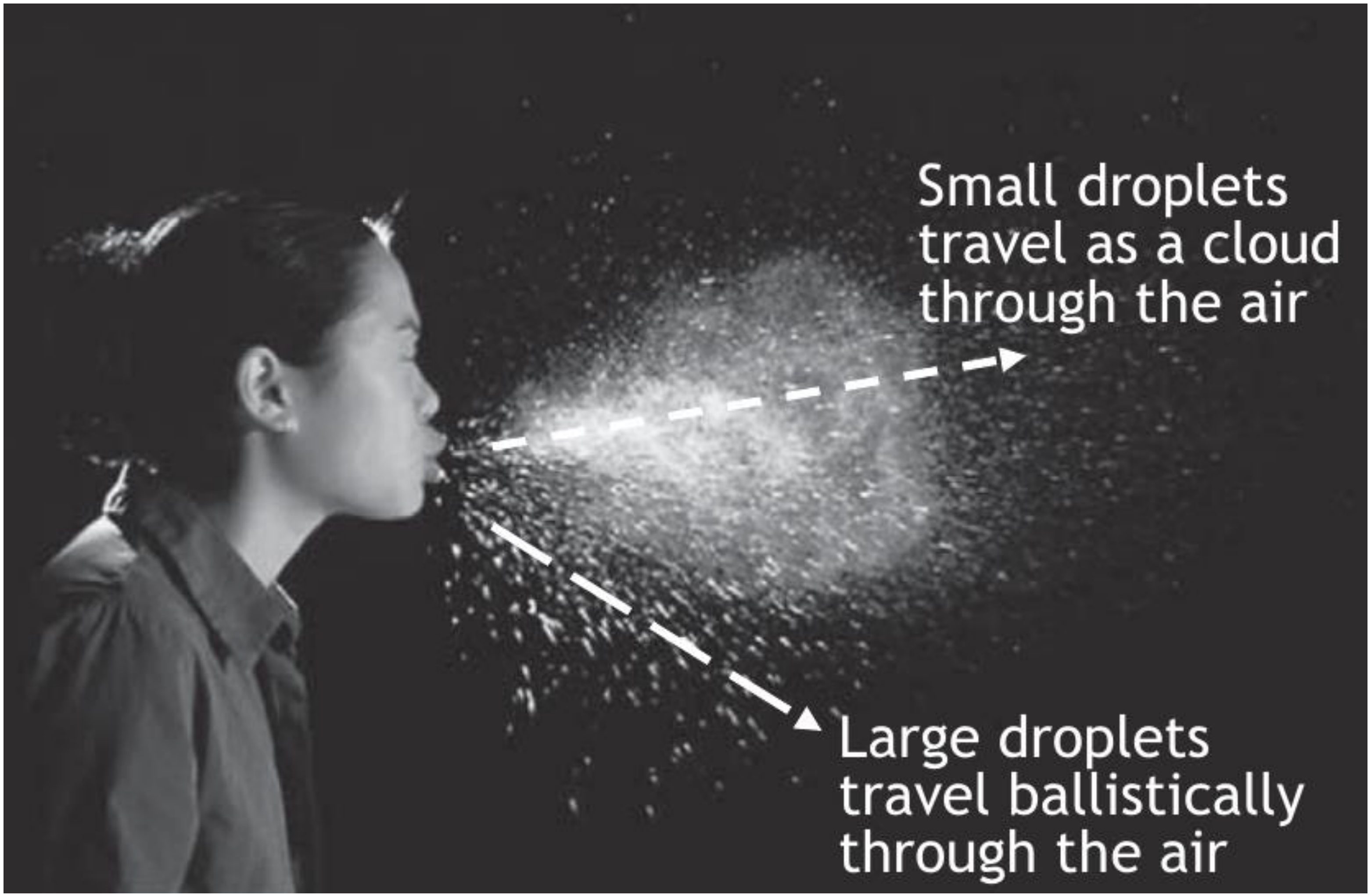
Outline

- Pathogen Transmission
- Mitigation Options
- Physical Modeling
- Numerical Methods
- Examples
- Reopening After the Crisis
- Conclusions and Outlook

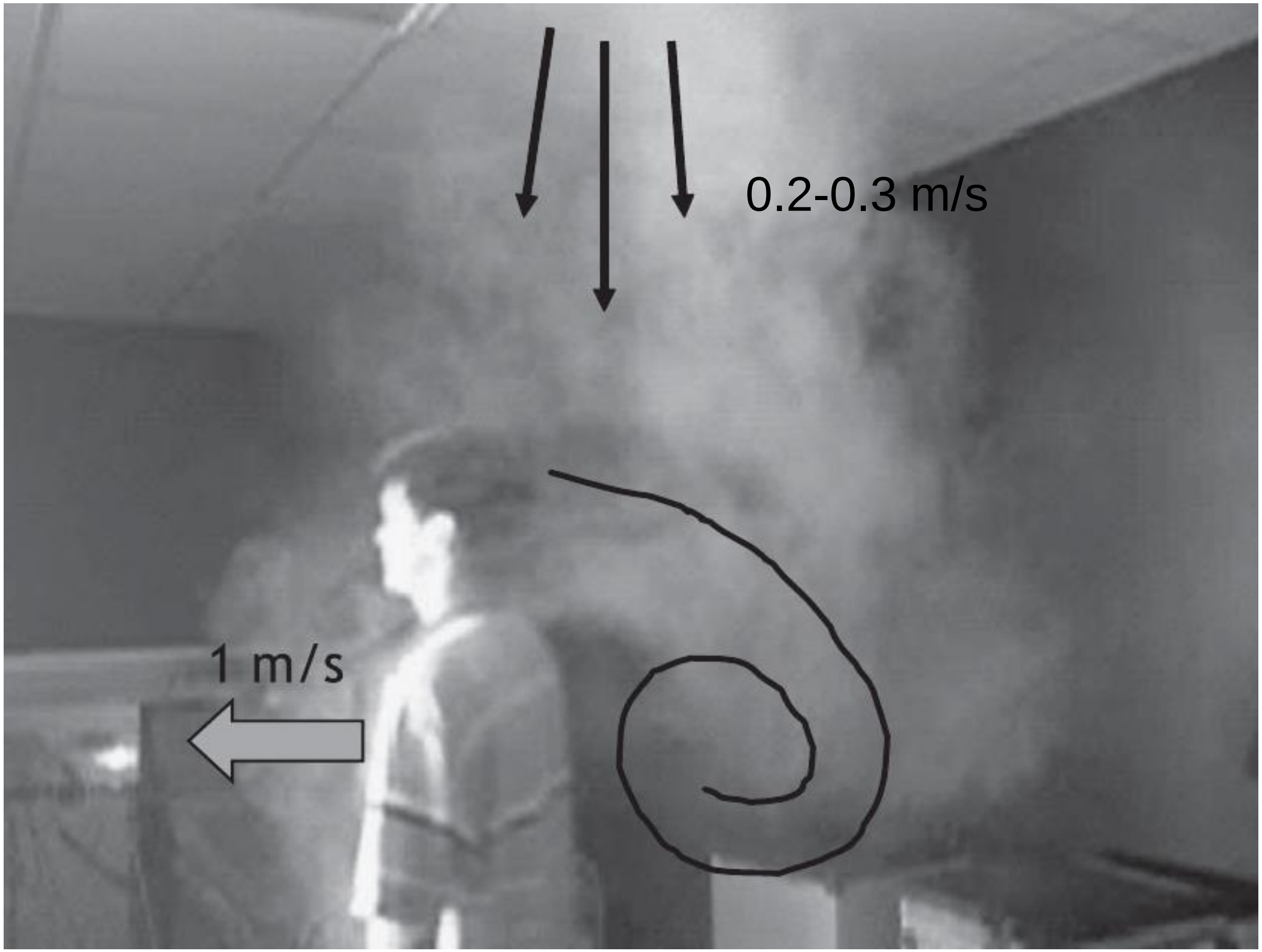
Pathogen Transmission

Virus Transmission: Sneezing



Virus Transmission: Breathing

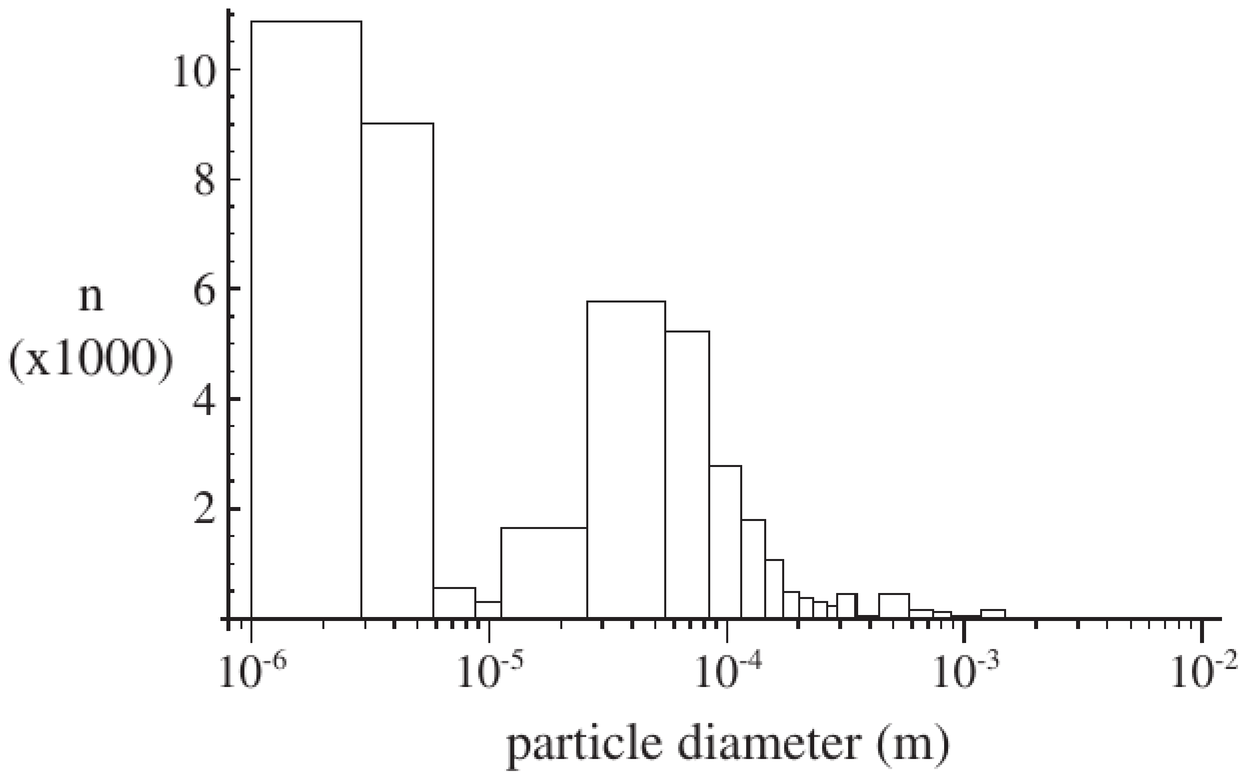
Virus Transmission: HVAC/Wakes



Virus Transmission

- Entering the Body: Nose, Mouth, Eyes
- Exiting the Body: Nose, Mouth
  - Sneeze, Cough, Shout, Sing, Speak, …
- ➔ Transmission Modes:
  - Person-to-Person : Large/Small Droplets
  - Person-Air-Person : Small Droplets
  - Person-Surface-Person : Large Droplets

Droplet Distribution When Coughing

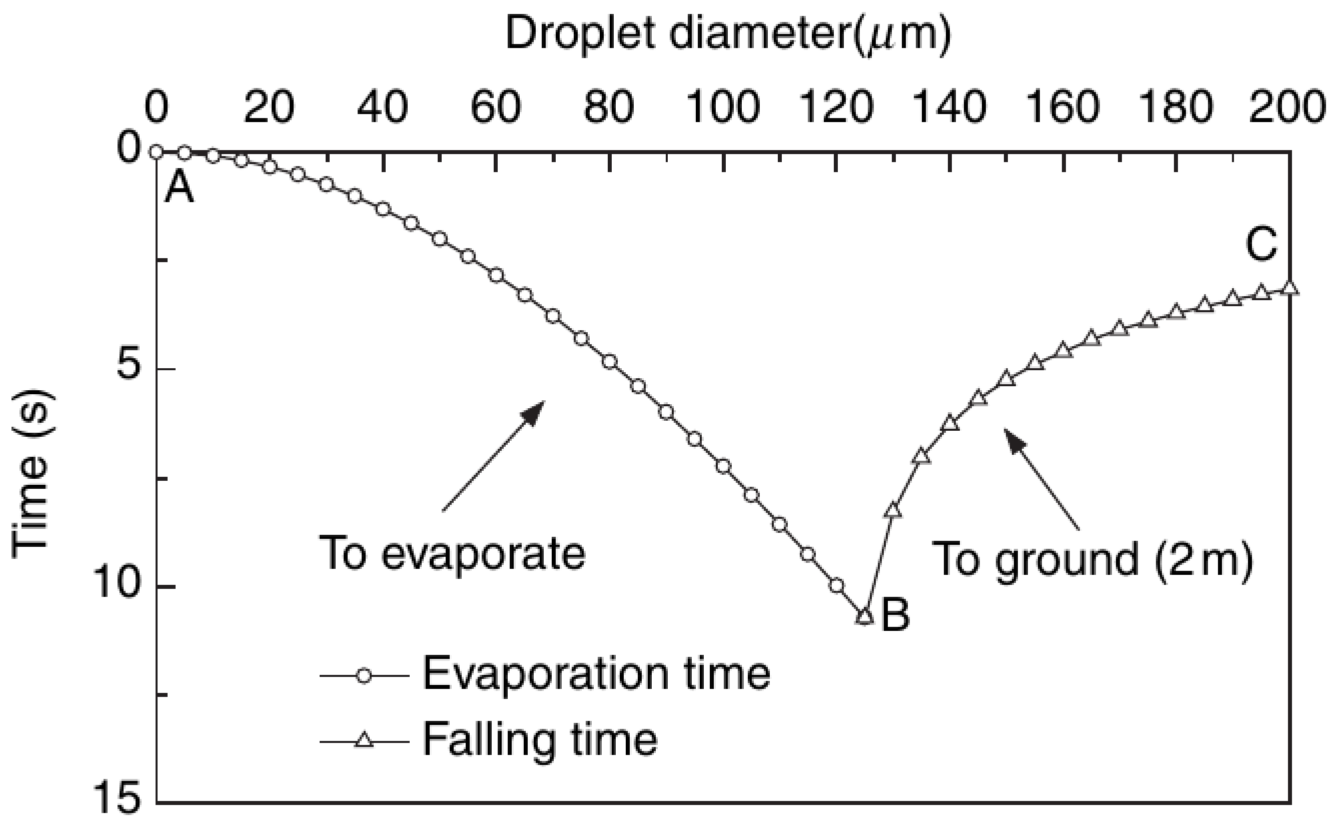


R.G. Loudon and R.M. Roberts - Droplet Expulsion from the Respiratory Tract; Am. Rev. Respir. Dis. 95, 3, 435–442 (1967).

Sink Velocities of Droplets in Air

Diameter [mm]	Sink Velocity [m/sec]	Reynolds-Nr.
1.0e+00	3.01e+01	1.99e+03
1.0e-01	3.01e-01	1.99e+00
1.0e-02	3.01e-03	1.99e-03
1.0e-03	3.01e-05	1.99e-06
1.0e-04	3.01e-07	1.99e-09

Droplet Evaporation



X. Xie, Y. Li, A.T.Y. Chwang, P.L. Ho, W.H. Seto - How Far Droplets Can Move in Indoor Environments – Revisiting the Wells Evaporation-Falling Curve; Indoor Air 17, 211-225 (2007). doi:10.1111/j.1600-0668.2006.00469.x

Viral Load / Infectious Dose

- Many Factors:
  - State of Immune Defenses of Victim
  - Timing of Viral Entry (All at Once, Piece by Piece)
  - Hair and Mucous in Nasal Vessels, ……
- Data From Biological Warfare Agents [Fra97]
  - Brucellosis 10-100
  - Q fever 1-10
  - Tularaemia 10-50
  - Smallpox 10-100
  - Viral Haemorrhagic Fevers 1-10
  - Tuberculosis 1 (!!)

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– Tuberculosis	1 (!!)

Cov-19: 100-200

Covid-19 Lifetime Outside the Body

- Air: 1-2 Hours
- Some Surfaces: 1-2 Days
- Some Variability With Humidity/Temperature
- Some Variability With UV/Sunlight Radiation
  - The More Sunlight, The Shorter The Lifetime

Infectivity of Covid-19

https://tinyurl.com/FAQ-aerosols

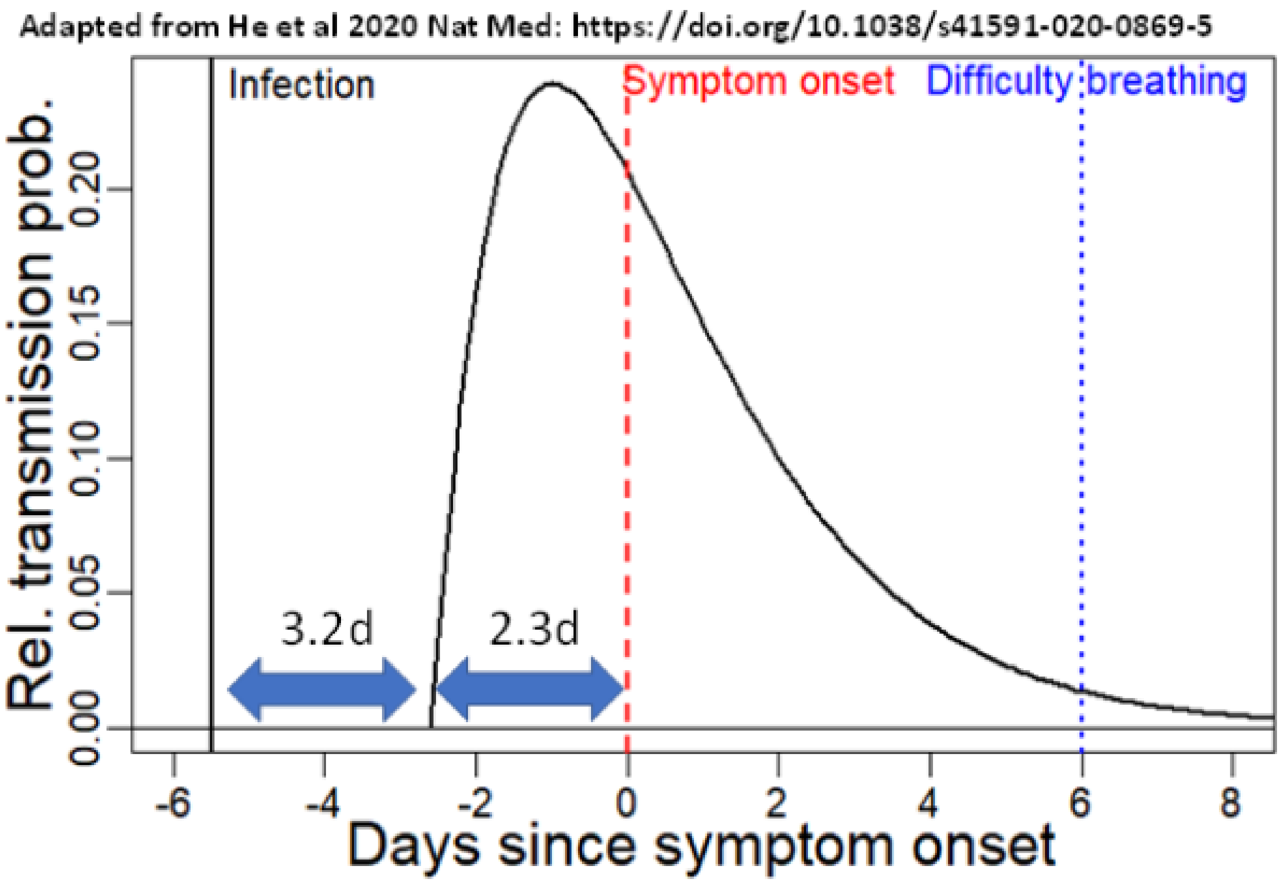

Figure: relative probability of transmission of SARS-CoV-2 from an infected individual as a function of time in the disease. The peak of infectiveness is just before the onset of symptoms. Reference: [Prof. A Marm Kilpatrick](), University of California-Santa Cruz.

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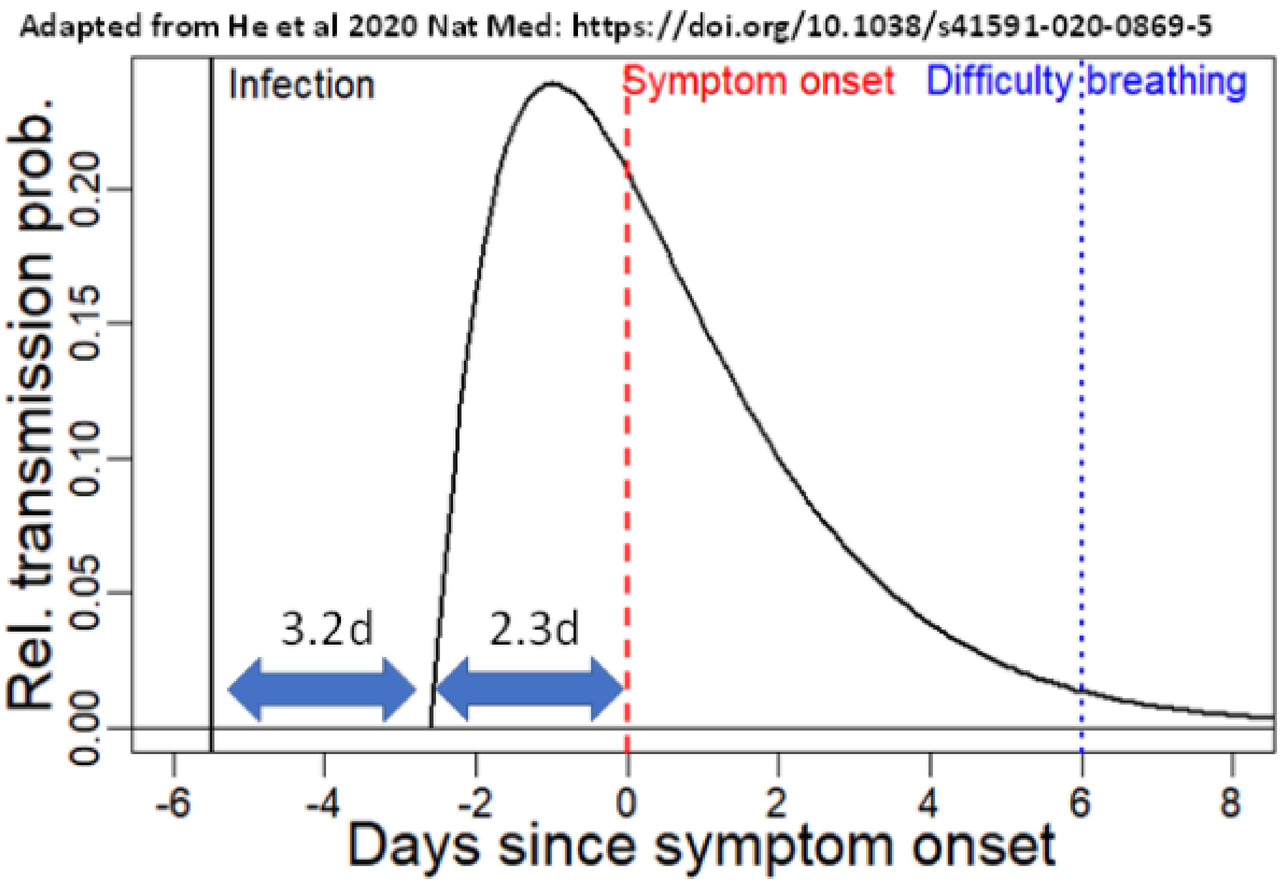

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Covid-19 vs. Influenza/Flu

- Max Yearly US Deaths From Influenza/Flu: 60K
  - No Lockdowns, No Preventive Measures

- US Deaths After 12-Months of Covid-19: 400K
  - With Lockdowns, With Preventive Measures

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- ➔ Covid-19 Orders of Magnitude More Deadly Than Influenza/Flu

Mitigation Options

Mitigation of Virus Transmission

- 2 Main Modes:
  - Large Droplets → `Spitting'
  - Small Droplets → `Smoke'

Procedure Measure	Large Droplets (spitting)	Small Droplets (cigarette smoke)	Person-Air-Person	Person-Surface-Person
2m/6ft Distance	✓	✓	✓	✓
Face Masks	✓	✓	✓	✓
Periodic Hand Cleaning	✓	✗	✓	✓
Plexiglass Shields	✓	✓	✓	✓
1-Way Person Traffic	✓	✓	✓	✓
2x Daily Cleaning	✓	✓	✓	✓
Nightly UV Cleaning	✓	✓	✓	✓
Maximize Fresh Air in HVAC	✗	✓	✓	✗
Hard UV Lamps in HVAC Ducts	✗	✓	✓	✗
HEPA Filters in HVAC Ducts	✗	✓	✓	✗
Upper Room UV Cleaning	✗	✓	✓	✗

Physics

Air: Navier-Stokes, Incompressible

Conservation of Mass, Momentum and Energy:

$$\nabla \cdot \mathbf{v} = 0$$

$$\rho \mathbf{v}_{,t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = \nabla \mu \nabla \mathbf{v} + \rho \mathbf{g} \beta (T_0 - T) + \mathbf{f}$$

$$\rho c_p T_{,t} + \rho c_p \mathbf{v} \cdot \nabla T = \nabla k \nabla T + s$$

ρ: Density	T: Temperature
$\mathbf{v}$: Velocity	c_p: Heat Capacitance
p: Pressure	k: Thermal Conductivity
μ: Viscosity	β: Expansion Coefficient
$\mathbf{g}$: Gravity	**f**,s: External Forces/Heat Sources

Equations for Diagnostics

- Age of Air

$$a_{,t} + \mathbf{v} \cdot \nabla a = 1$$

- Pathogen Concentration

$$c_{,t} + \mathbf{v} \cdot \nabla c = \nabla d_c \nabla c + s_\mathrm{c}$$

- UV Irradiation

$$I_{,t} + \mathbf{v} \cdot \nabla I = \nabla d_I \nabla I + s_\mathrm{I}$$

- …

Particle Motion and Temperature

- Velocity and Position

$$\rho_p \frac{\pi d^3}{6} \frac{d\mathbf{v}_{\mathrm{p}}}{dt} = \mathbf{D} \quad ; \quad \frac{d\mathbf{x}_{\mathrm{p}}}{dt} = \mathbf{v}_{\mathrm{p}}$$

- Temperature

$$\rho_p c_{pp} \frac{\pi d^3}{6} \frac{dT_{\mathrm{p}}}{dt} = Q$$

Momentum Transfer

- Drag Force of Each Particle

$$D_i = \frac{\pi d^2}{4} c_d \frac{1}{2} \rho \left| \mathbf{v} - \mathbf{v}_{\mathrm{p}} \right| \left(\mathrm{v}_{\mathrm{i}} - \mathrm{v}_{\mathrm{pi}} \right)$$

- Drag Coefficient and Reynolds-Number

$$c_d = \max\left(0.1, \frac{24}{\mathrm{Re}} \left(1 + 0.15\,\mathrm{Re}^{0.687} \right) \right) \quad ; \quad \mathrm{Re} = \frac{\rho \left| \mathbf{v} - \mathbf{v}_{\mathrm{p}} \right| d}{\mu}$$

Heat Transfer

- Heat Flux For Each Particle

$$Q = \pi d^2 \left[h\left(T - T_p\right) + \sigma^{\times}\left(T^4 - T_p^4\right) \right]$$

- Film Coefficient, Nusselt- and Prandtl-Number

$$h = \frac{k \cdot Nu}{d} \quad ; \quad Nu = 2 + 0.459 \, \mathrm{Pr}^{0.333} \, \mathrm{Re}^{0.55}; \quad \mathrm{Pr} = \frac{c_p k}{\mu}$$

Numerics

Basic Elements of Solver (1)

- Spatial Discretization: Unstructured Grids
  - Arbitrary Geometries
  - Adaptive Refinement
- Spatial Discretization: Simple FEM
  - One-Type Element Code: Speed and Simplicity
  - Use Tetrahedra Even in Boundary Layers
- Temporal Discretization: Explicit Advection
  - Physically Interesting Scales ➔ Need Accuracy
  - 1-Step Schemes: Moving Grids, Adaptive Refine/Remesh
- Temporal Discretization: Implicit Viscous/Pressure
  - Low-Storage Iterative Solvers

Basic Elements of Solver (2)

- Steady Results: Independent of Timestep
  - Confidence
  - Convergence Study Possible
- Edge-Based Data Structures
  - Reduction of Indirect Addressing
  - Reduction in Flops
- Extensive Renumbering
  - Avoidance of Cache-Misses
  - Shared-Memory Parallel Machines
- Preconditioning:
  - Diagonal: Isotropic Grid
  - Linelets: Stretched (RANS) Grid

Temporal Discretization: Δ-Scheme (1)

$$\mathrm{v}^{n+1} = \mathrm{v}^n + \Delta\mathrm{v}^a + \Delta\mathrm{v}^p = \mathrm{v}^* + \Delta\mathrm{v}^p$$

- Advective / Diffusive Prediction: $\mathrm{v}^n \to \mathrm{v}^*$

$$\left[\frac{1}{\Delta t} - \theta\nabla\mu\nabla\right](\mathrm{v}^* - \mathrm{v}^n) + \mathrm{v}^n \cdot \nabla\mathrm{v}^n + \nabla p^n = \nabla\mu\nabla\mathrm{v}^n$$

- Pressure Correction: $p^n \to p^{n+1}$

$$\nabla \cdot \mathrm{v}^{n+1} = 0$$

$$\frac{\mathrm{v}^{n+1} - \mathrm{v}^*}{\Delta t} + \nabla(p^{n+1} - p^n) = 0$$

$$\Rightarrow \Rightarrow \quad \nabla^2(p^{n+1} - p^n) = \frac{\nabla \cdot \mathrm{v}^*}{\Delta t}$$

Temporal Discretization: Δ-Scheme (2)

- Velocity Correction: $\mathrm{v}^* \rightarrow \mathrm{v}^{n+1}$

$$\mathrm{v}^{n+1} = \mathrm{v}^* - \Delta t \nabla (p^{n+1} - p^n)$$

Remarks:

- Residuals of Pressure Correction Vanish for Steady-State
  - ➔ Results Not Depend on Projection Scheme
  - ➔ Results Not Depend on Δt

Particle Motion and Temperature

- Position, Velocity and Temperature: ODEs of Type

$$\frac{du}{dt} = r(\mathbf{x}, \mathbf{u}_f, u, t)$$

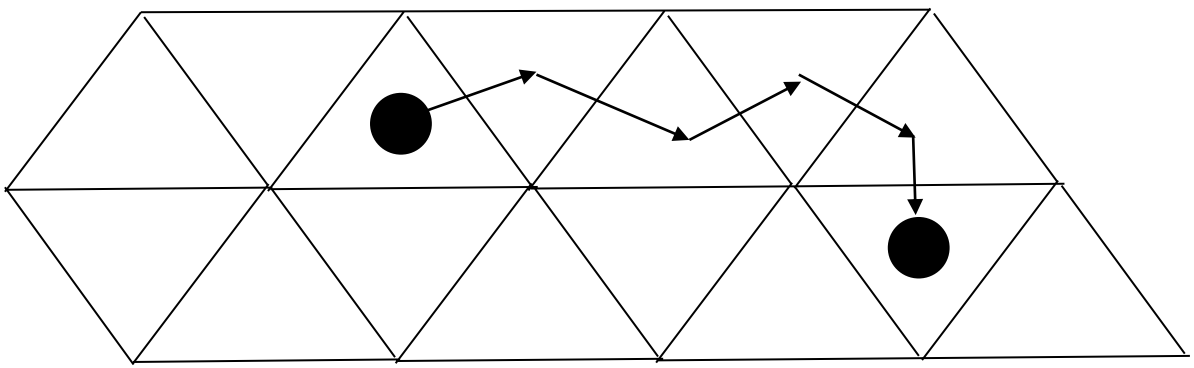
- Integrated Explicitly; Typically: 4th Order Runge-Kutta

$$u^{n+i} - u^{n+i-1} = \alpha_i \Delta t \cdot r(\mathbf{x}, \mathbf{u}_f, u^{n+i-1}, t^{n+i-1})$$

$$\alpha_i = \frac{1}{k - i + 1}$$

Particle Tracking

- Need: Flow Variables At Location of Particle
- ➔ Need Host Element for Each Particle



- Initialization: Bins + Near-Neighbour Search
- Incremental: Near-Neighbour Search
  - Vectorized and Parallelized for OMP
  - Also Running in MPI

UV Radiation

- Irradiation Function of Distance/Angle

$$s_I \sim \frac{1}{r^2}$$

- Shading Possible ➔ Ray-Tracing
  - From Element (Gauss-Points) to UV Lamp

Raytracing With FEM Grids

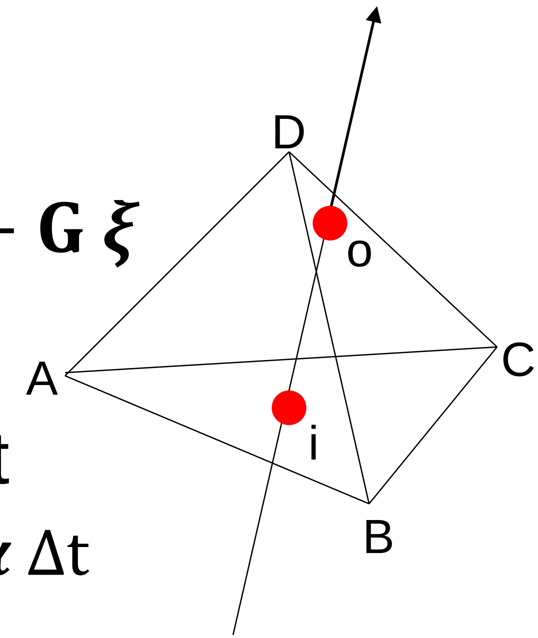
- Any Point P:

  $$\mathbf{x}_P = \mathbf{x}_A + \mathbf{x}_{BA}\,\xi + \mathbf{x}_{CA}\,\eta + \mathbf{x}_{DA}\,\zeta = \mathbf{x}_A + \mathbf{G}\,\boldsymbol{\xi}$$

- ➔ $\boldsymbol{\xi} = \mathbf{G}^{-1}\mathbf{x}_{PA}$
- Given Input Location, Obtain Output

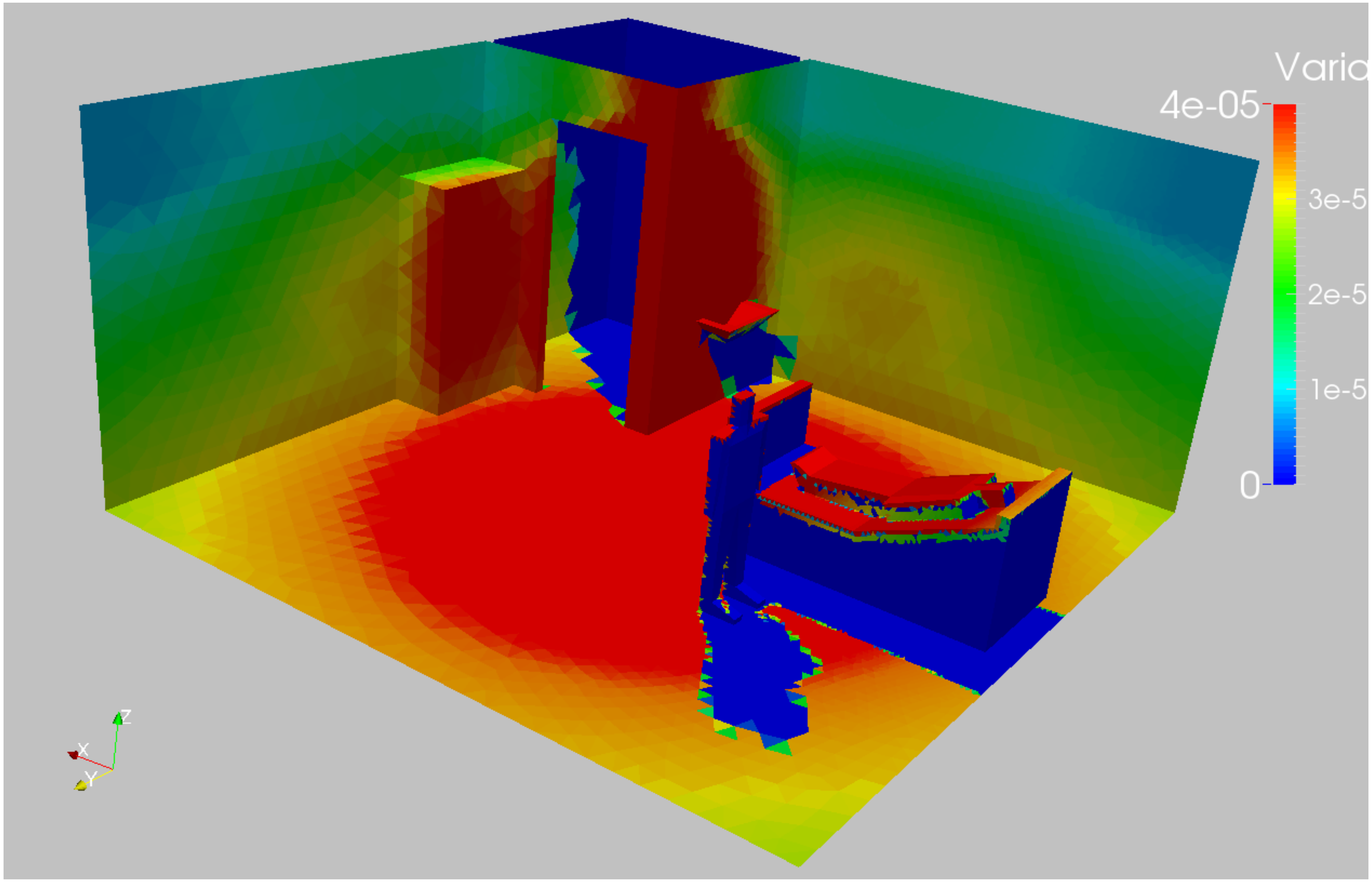
  $$\boldsymbol{\xi}\mathrm{o} = \boldsymbol{\xi}\mathrm{i} + \mathbf{G}^{-1}\Delta\mathbf{x} = \boldsymbol{\xi}\mathrm{i} + \mathbf{G}^{-1}\mathbf{v}\Delta\mathrm{t} = \boldsymbol{\xi}\mathrm{i} + \boldsymbol{\alpha}\,\Delta\mathrm{t}$$

- Output Faces: $\xi_o=0$
- Get $\min(\Delta t_i)$, $\Delta t_i > 0$ ➔ Neighbour Element



J. Favre and R. Löhner – Ray Tracing With a Space-Filling Finite Element Mesh; IJNME 37, 3571-3580 (1994)

Ceiling UV In Hospital Room



Coupling of CFD and CCD

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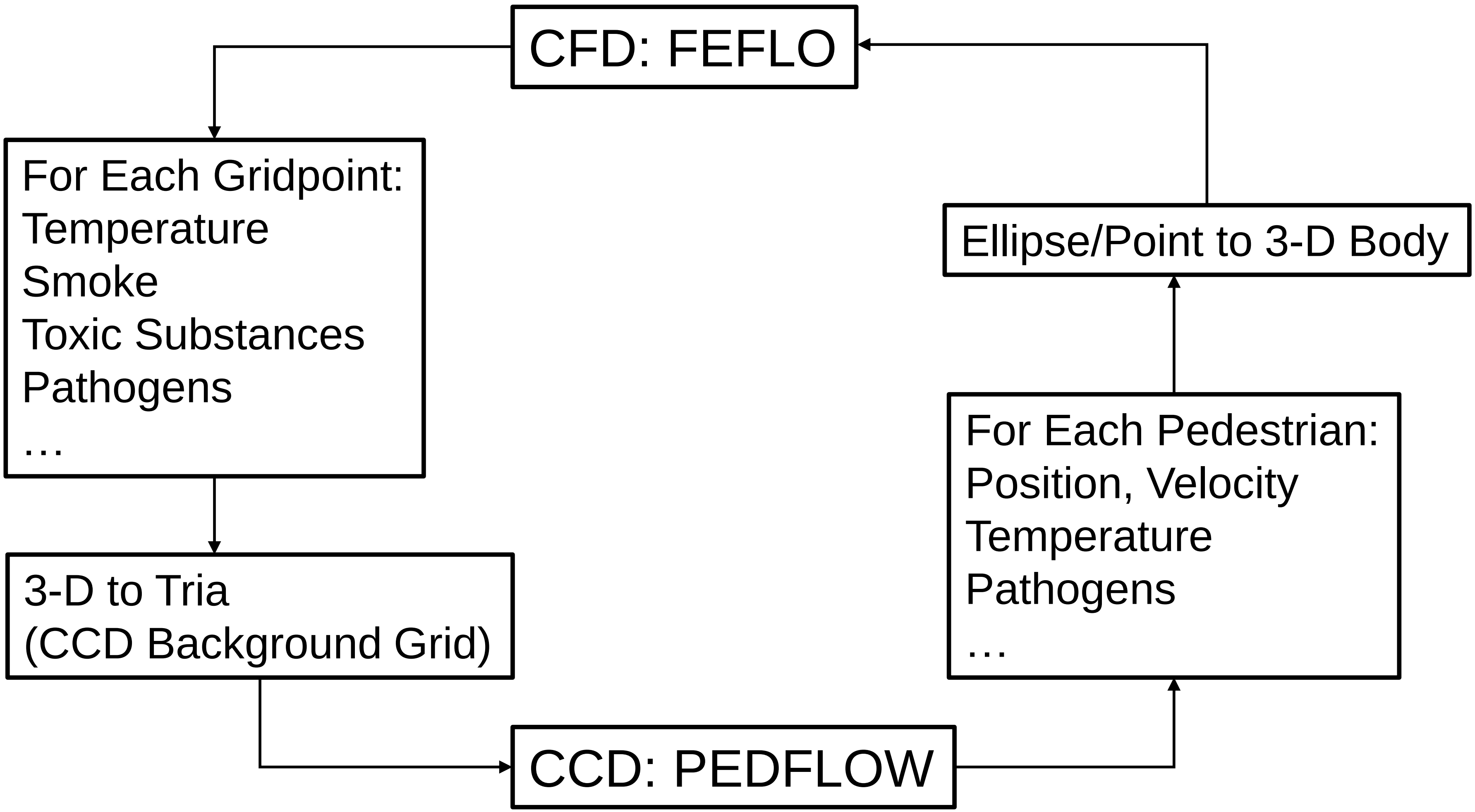
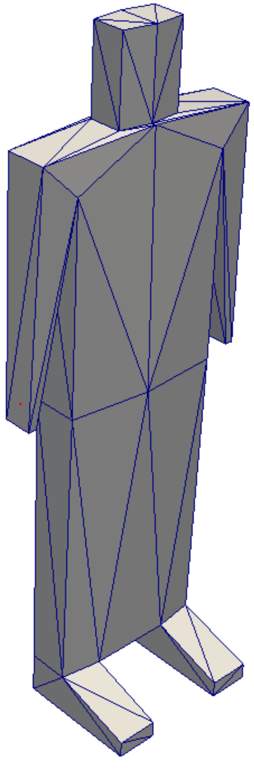


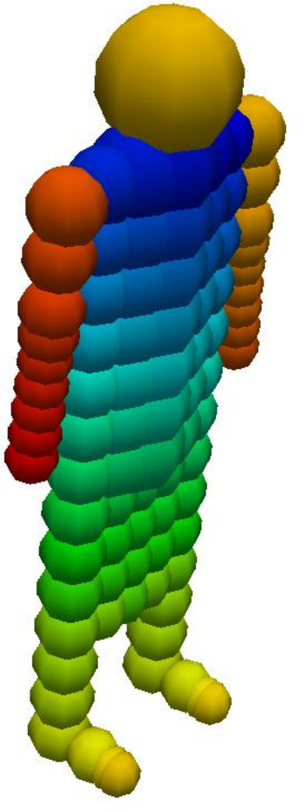
CCD → CFD

- Several Options Possible
- Body Fitted
  - Transcribe Discrete Surface from CCD → CFD / Merge
  - Move/Smooth/Remesh CFD Mesh
- Embedded
  - Transcribe Discrete Surface from CCD → CFD
  - Obtain Intersections With Edges/New BC
- Immersed
  - Transcribe Discrete Volume from CCD → CFD
  - Obtain CFD Points Inside CCD Domain/New BC

CCD → CFD



As Tet-Mesh



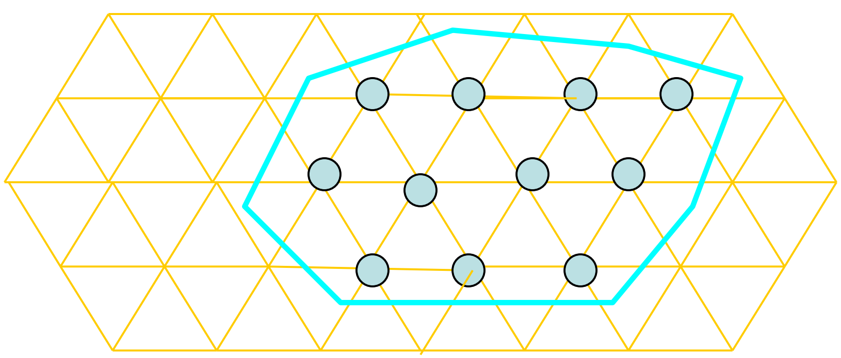
As Spheres

Immersed Bodies

Immersed Body: Options

- Desired: In Body Region: $v_f = w_b$
- **Kinematic**: Impose: $v_f = w_b$
- **Kinetic** : Use Force: $f = c_0 (v_f - w_b)$ [Goldstein]
- **Kinetic/Kinematic**: [Mohd-Yusof]:

$$\mathbf{M}\frac{\Delta \mathbf{v}_i}{\Delta t} = \mathbf{r}_i + \mathbf{f}_i \quad \Rightarrow \quad \mathbf{f}_i = \mathbf{M}\frac{\mathbf{w}_i^{n+1} - \mathbf{v}_i^n}{\Delta t} - \mathbf{r}_i$$



Immersed Body: Extensions

- Kinematic:
  - Extend to Higher Order [Balaras]
  - Same as for Embedded
- Kinetic:
  - Use Lagrange Multipliers [Glowinsky]

Search for Points in Bodies: Options

- Option 1:
  - Store CFD Points in Bin/Octree
  - Loop Over Immersed Body Elements
    - Get Bounding Box
    - Get CFD Points in Bounding Box
    - Detailed In/Out Analysis
- Option 2:
  - Store Immersed Body Elements in Bin/Octree
  - Loop Over CFD Points
    - Get Immersed Body Element in Region
    - Detailed In/Out Analysis

Pedestrian Motion

Discrete Models

- Any Pedestrian Flow Simulation:
  - Global Movement: Strategic, Tactical
  - Local Movement: Operational
- Global Movement
  - Targets (Regions/Lines/…) ➔ Will Force
- Local Movement
  - Collision Avoidance
    - Social Force/ Contact Models
  - Local Geometry Info
    - Walls, Paths, Roughness, …

PEDESTRIAN MOTION

- Newton's Law:

$$m\,\mathbf{v}_{,t} = \mathbf{f}$$

$$\mathbf{x}_{,t} = \mathbf{v}$$

- m: Mass
- **v**: Velocity
- **x**: Position
- **f**: Sum of All Forces
- Modeling Effort: **f**

PEDESTRIAN FORCES

- Internal Forces
  - Will Force (Get There (in Time))
  - Pedestrian Collision Avoidance Forces: Intermediate Range
  - Pedestrian Collision Avoidance Forces : Near Range
  - Wall/Obstacle (Collision) Avoidance Forces
- External Forces
  - Contact: Other Pedestrians
  - Contact: Walls/Obstacles
- …

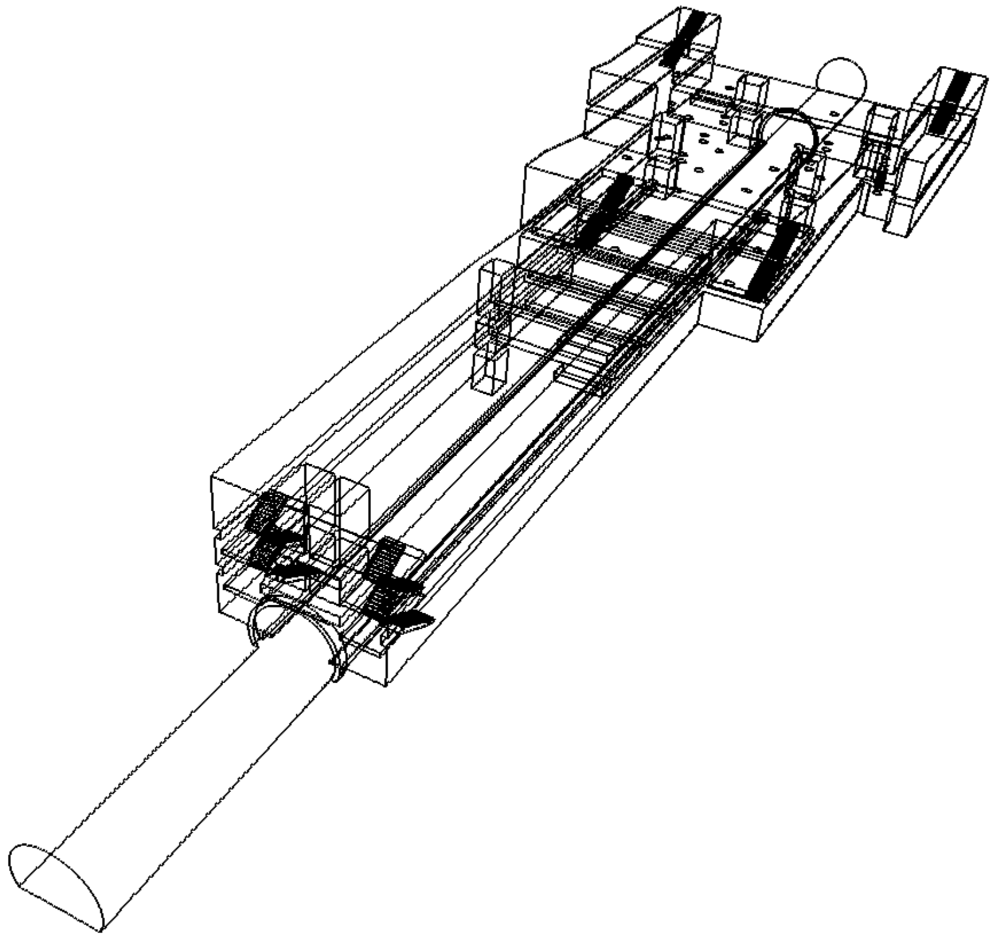
PEDFLOW

- Mixture of Agent-Based and Social Force Model
- Forces Via by Minimal Set of Well-Defined Parameters
  - Relaxation Time (Fitness)
  - Desired Velocity
  - Pushiness (Distraction)
- Strategic and Tactical: Desired Locations/Time
- Operational: Local Collision Avoidance
- Background Grid for Geometry/Spatial Search
- Edge-Based Data Structures for Pedestrians

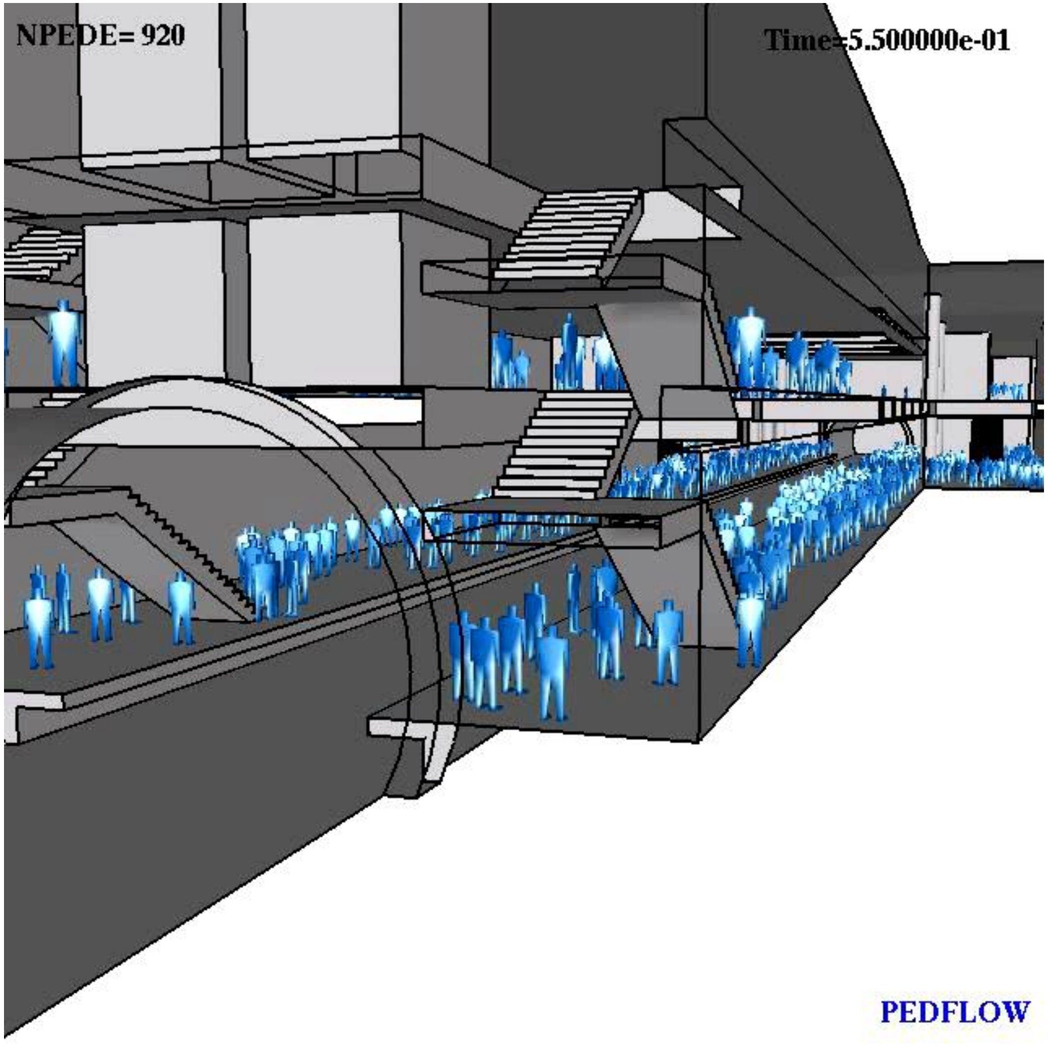
R. Löhner – On The Modeling of Pedestrian Motion;
Appl. Math. Mod. 34, 2, 366-382 (2010).

Madrid Metro Station

- 3/11/2004 Attack
- Did Blast Analysis
- Follow-Up



Madrid Bomb Attack



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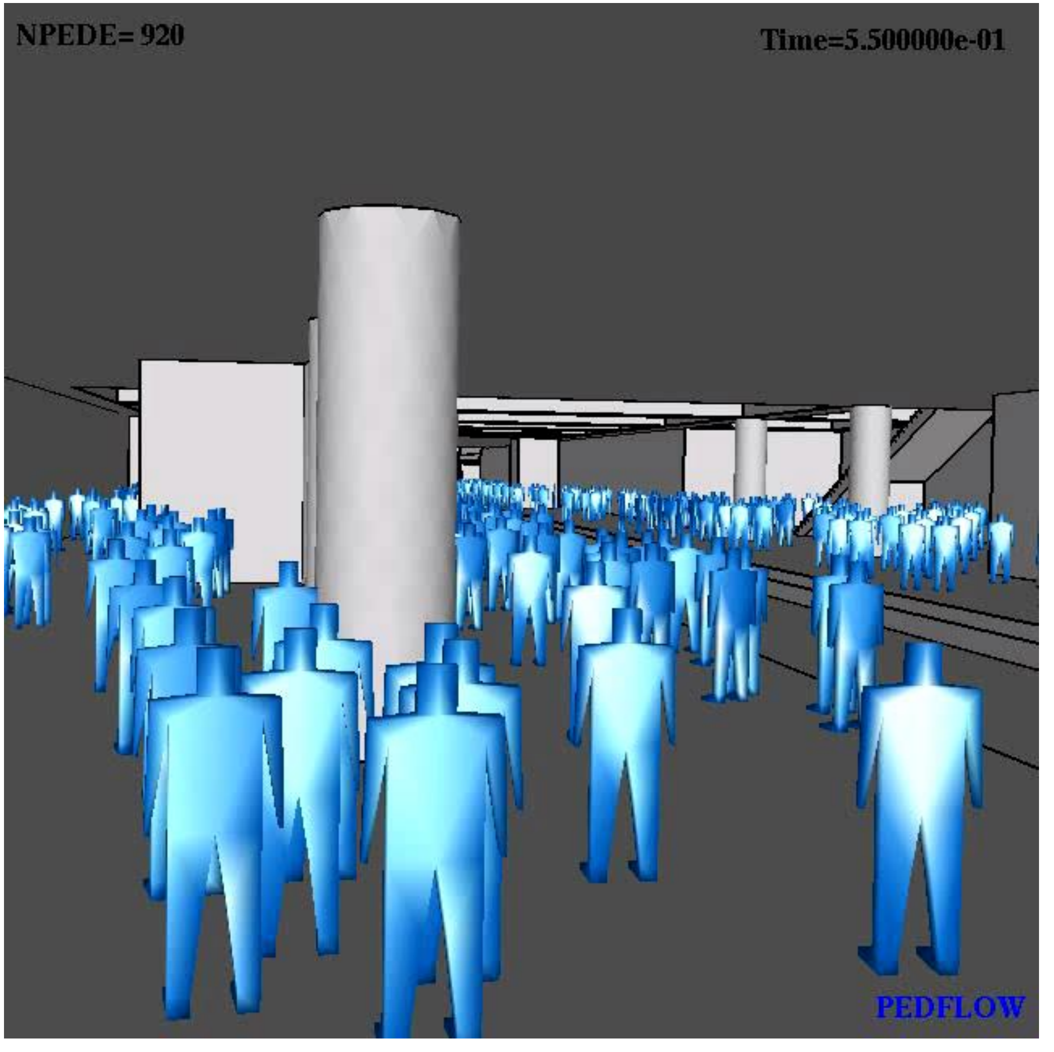
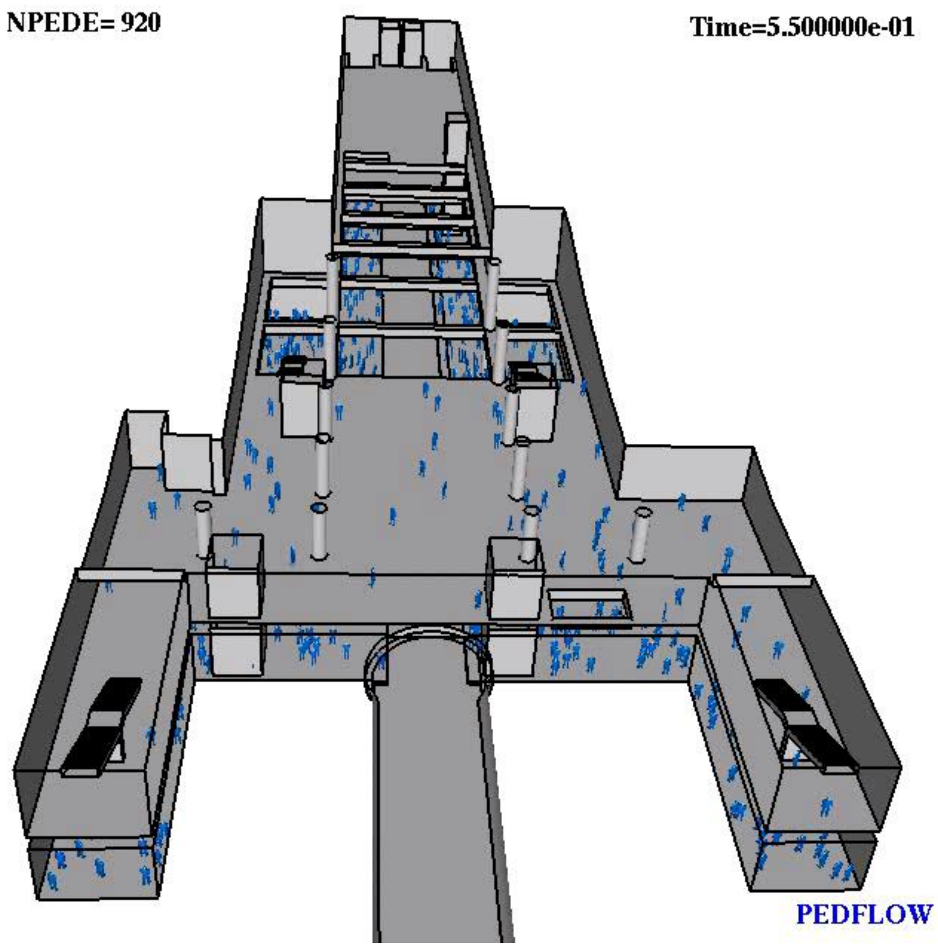
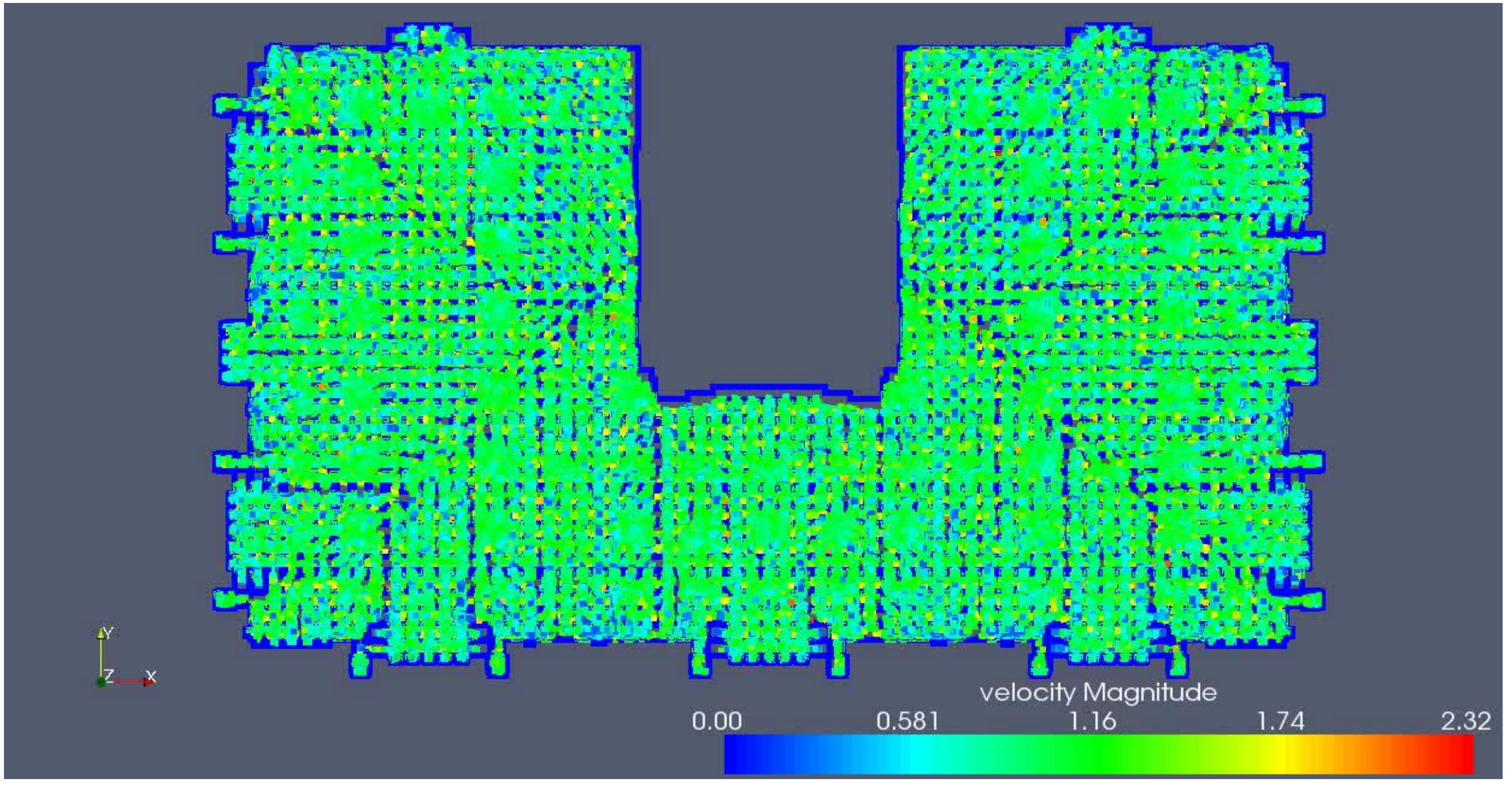


Madrid Bomb Attack

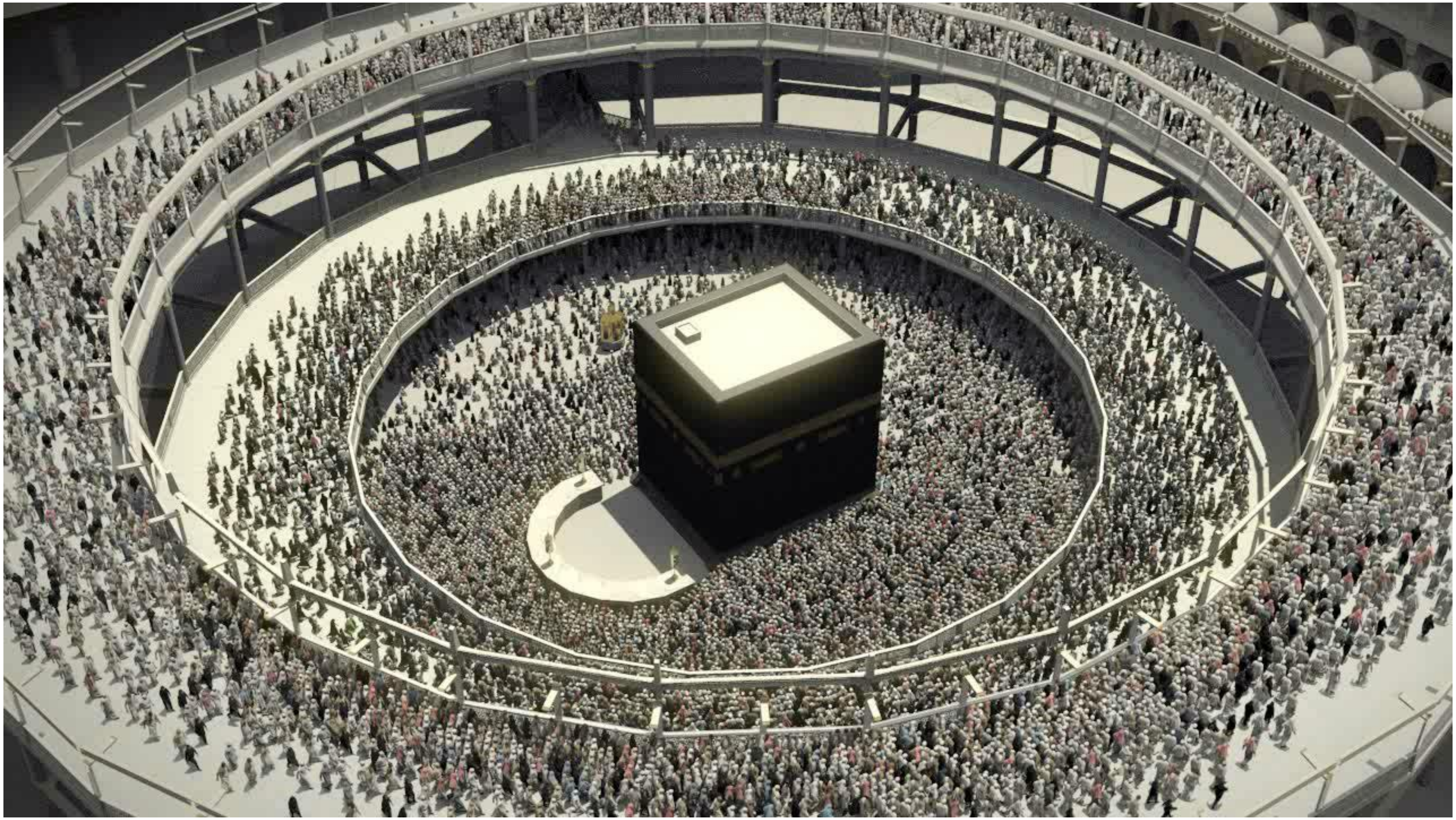


Evacuation from Medina Mosque

- Over 10^5 Pedestrians
- Run on Laptop

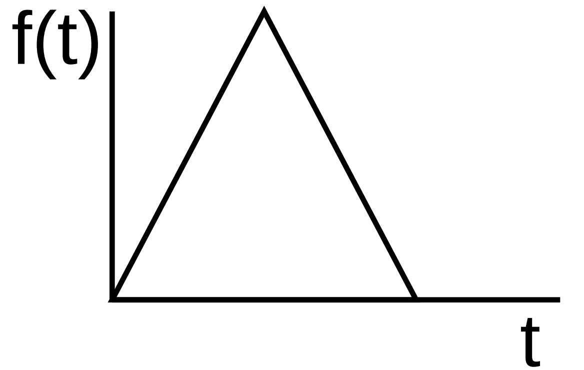


Rendering via 3-D Studio Max



Examples

Examples (1)

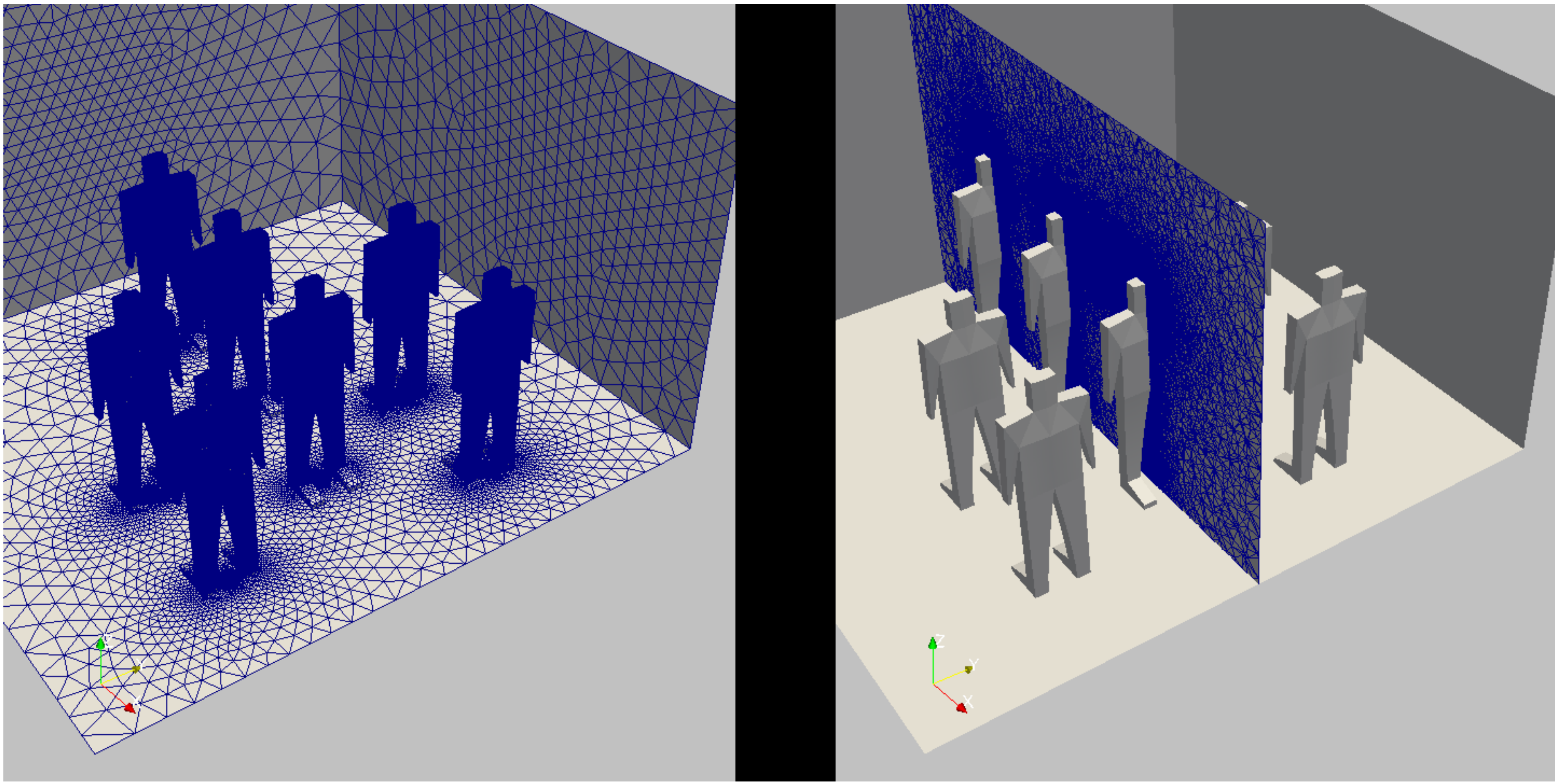


- Physics and Numerics: FEFLO
- Flow Initialization:
  - Ambient: Quiescent, 20^{o}C
  - Sneeze: Spherical Region, r=5cm, Near Mouth
  - V=5f(t) [m/sec] ; T=20+(37-20)f(t) [^{o}C]
- Particle Initialization
  - 4 Size Groups; Released Every 0.005 sec for 0.1 sec
  - V=5 m/sec ; T=37^{o}C

Examples (2)

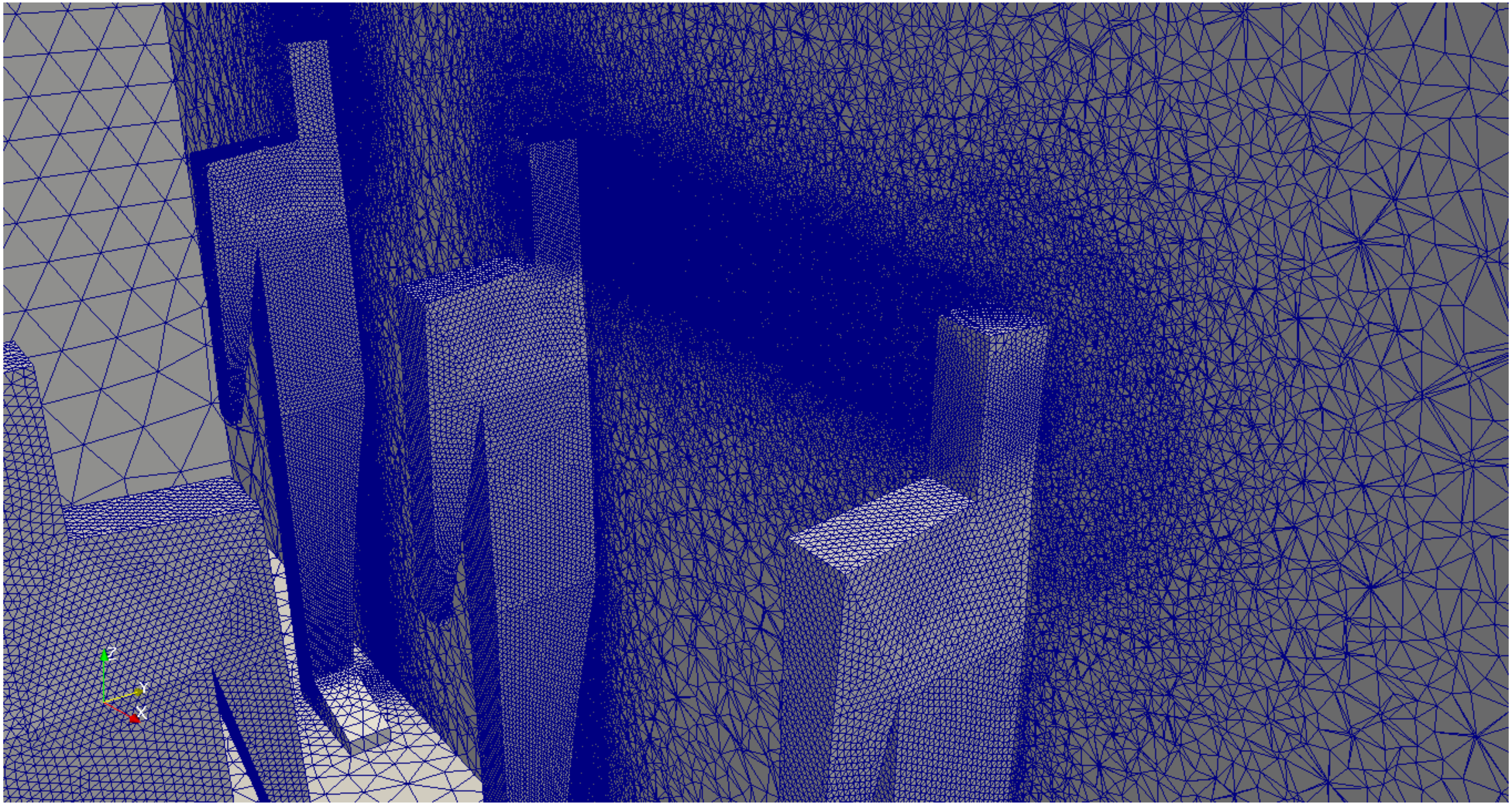
- 3 Different Timescales, Depending on Size
  - $O(10^0)$ sec: Fast, Ballistic Drop of Large d=1.0 mm Particles
  - $O(10^1)$ sec: Slower Drop of Intermediate d=0.1 mm Particles
  - $O(10^2)$ sec: Transport of Small d<0.01 mm Particles Through Air
- HVAC Systems:
  - Good Mixers
  - Complex Flowfields
  - ➔ Condusive to Pathogen Transmission

Sneezing in TSA Queues



12.47 Mels

Sneezing in TSA Queues



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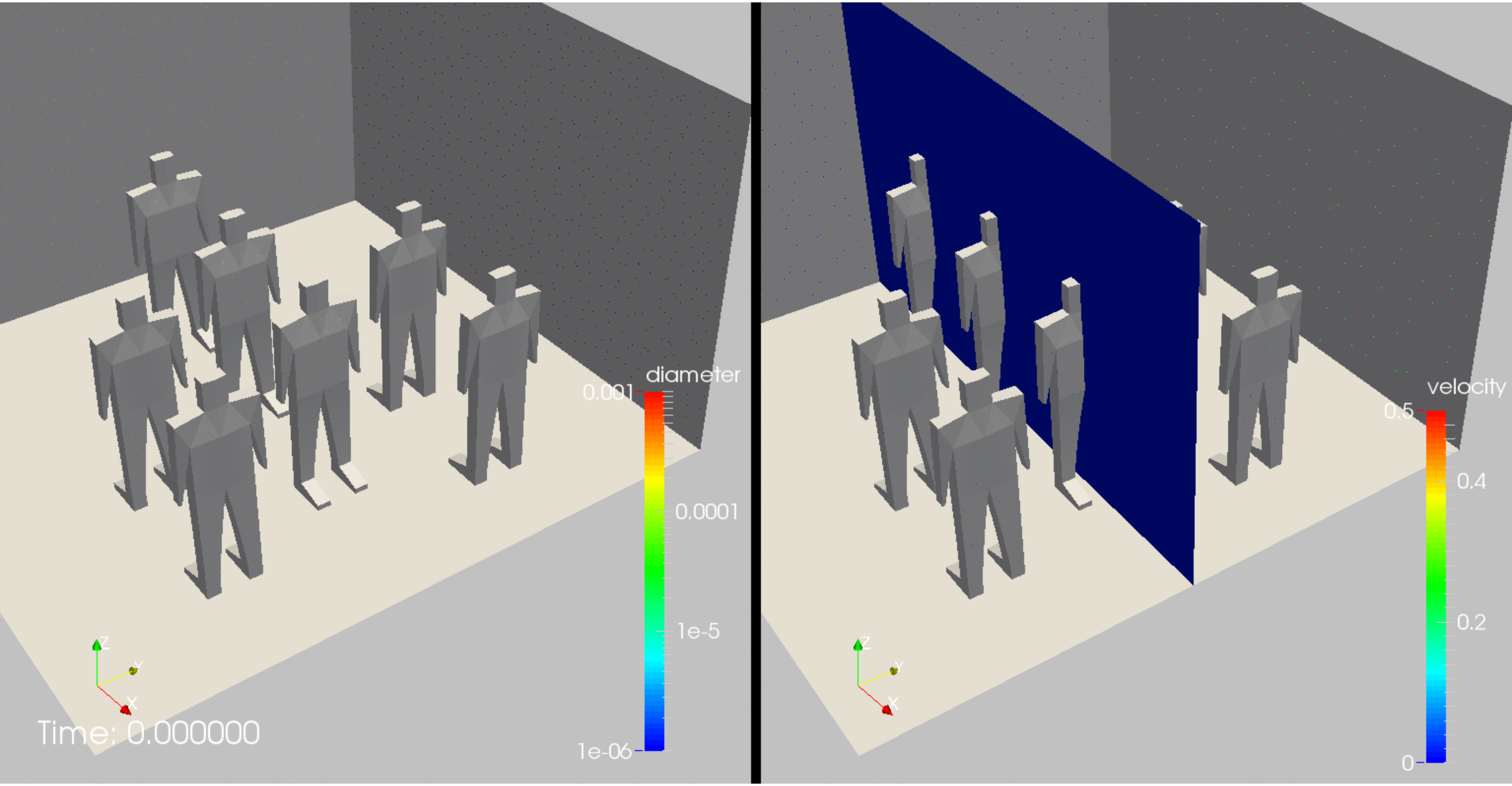
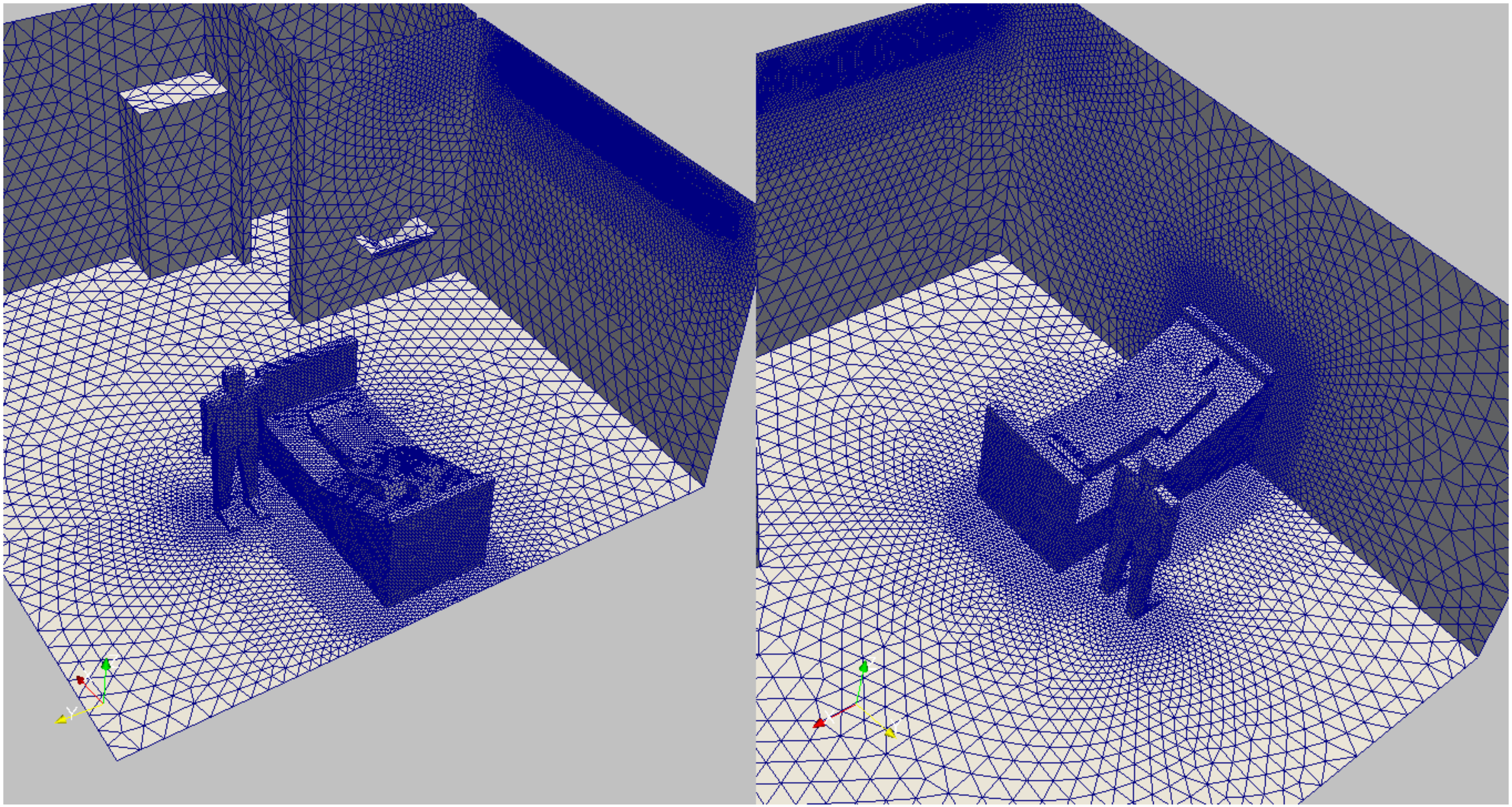
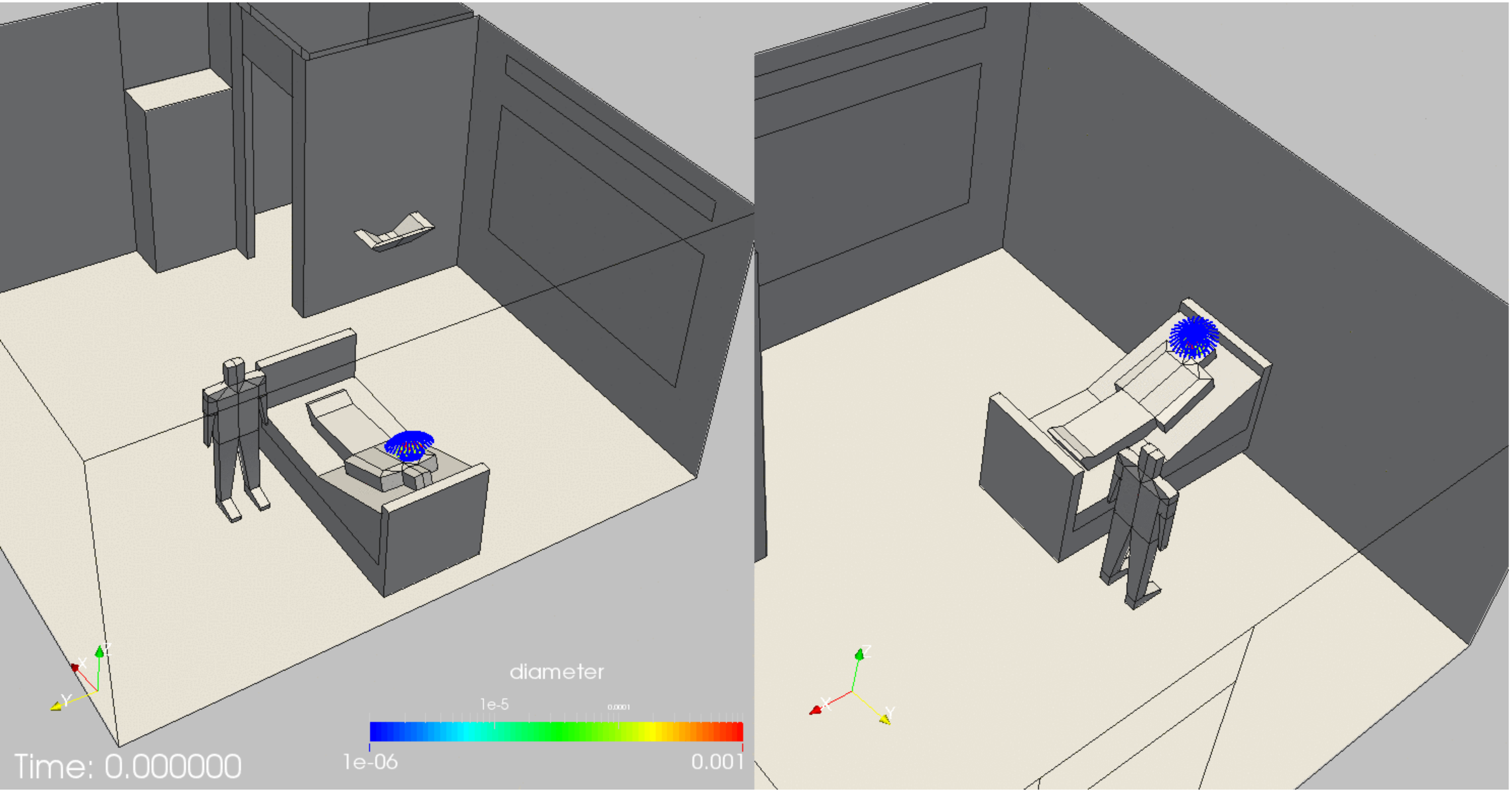


Sneezing in Hospital Room

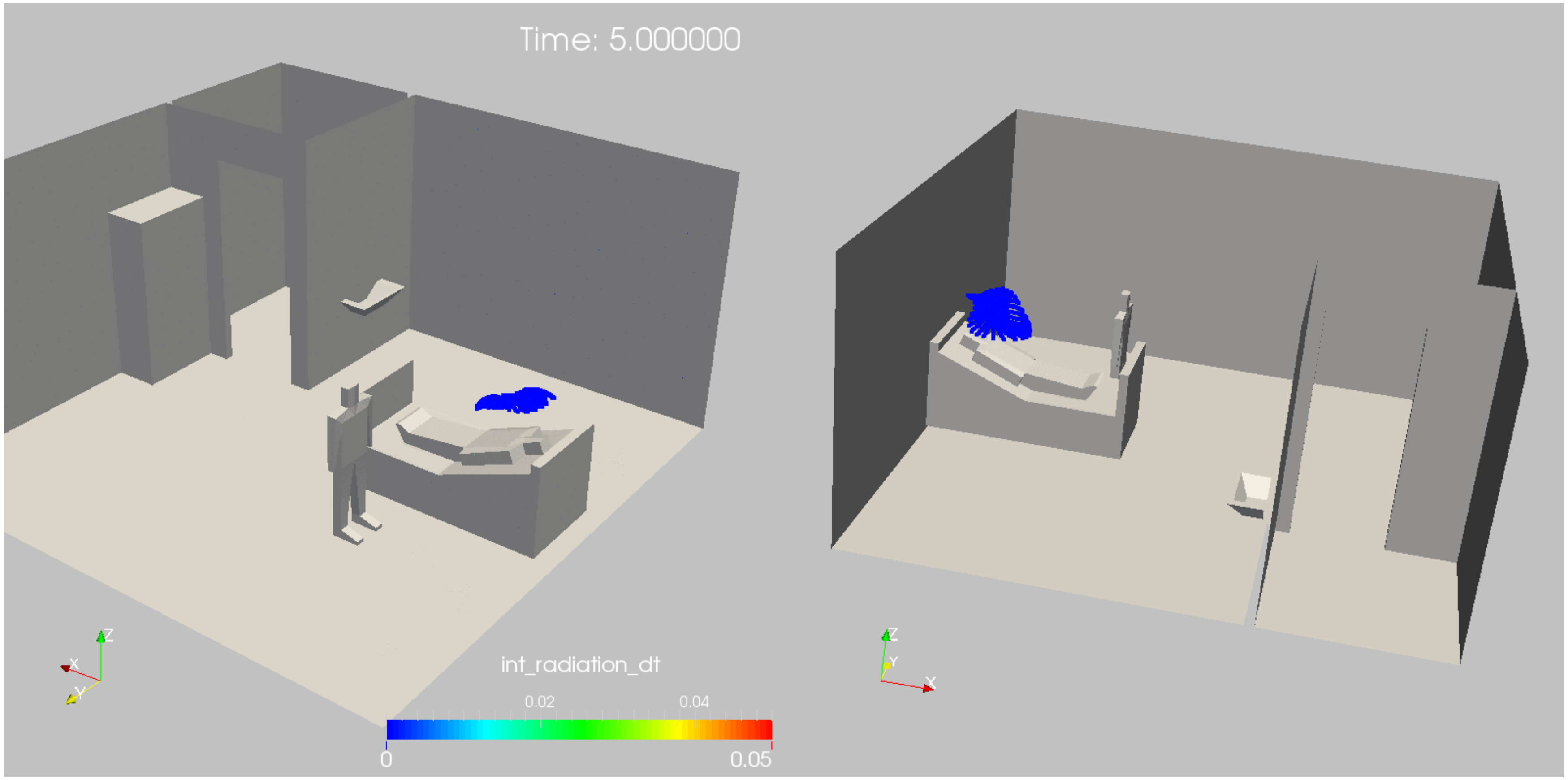


2.25 Mels

Sneezing in Hospital Room



Hospital Room With UV Lamp



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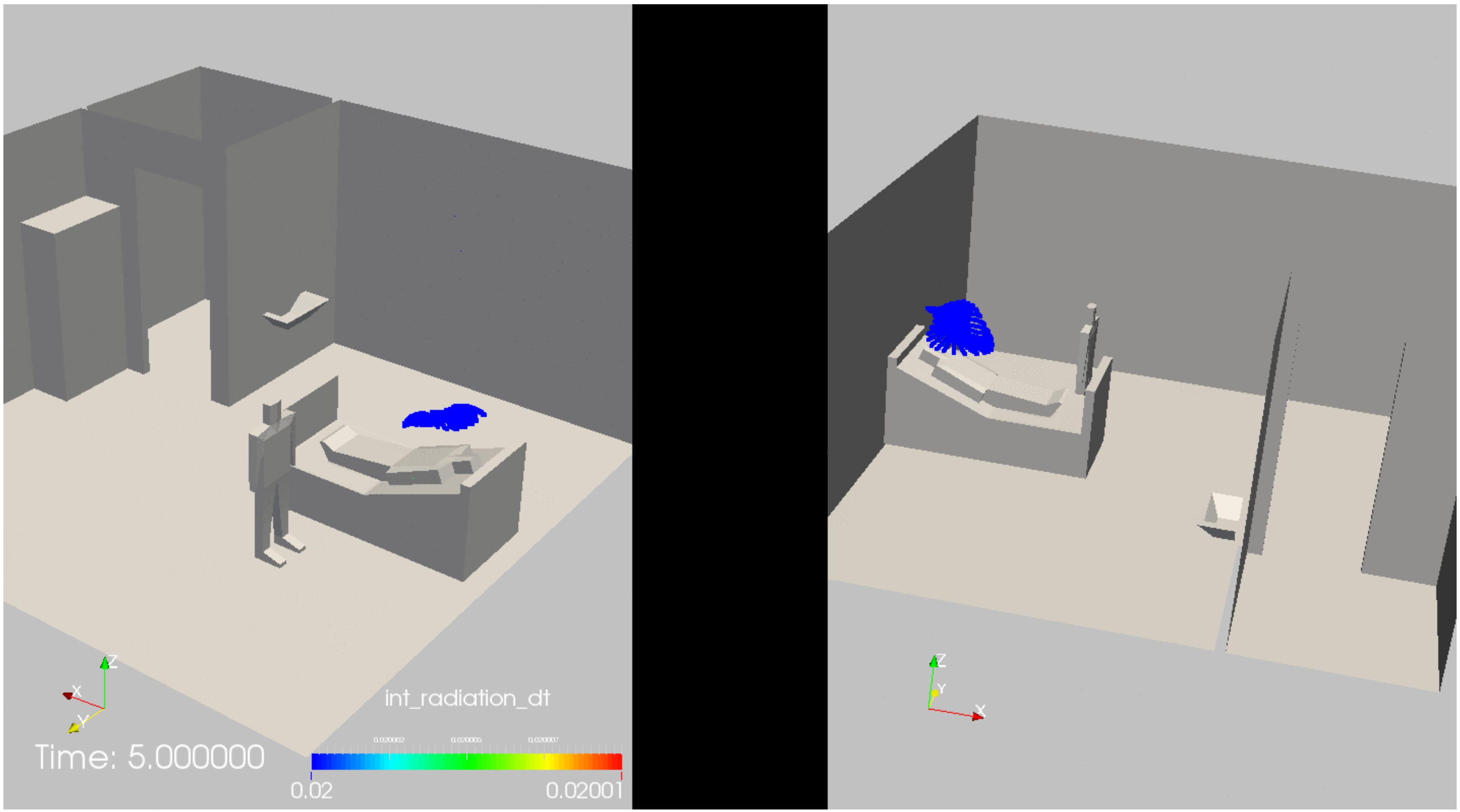
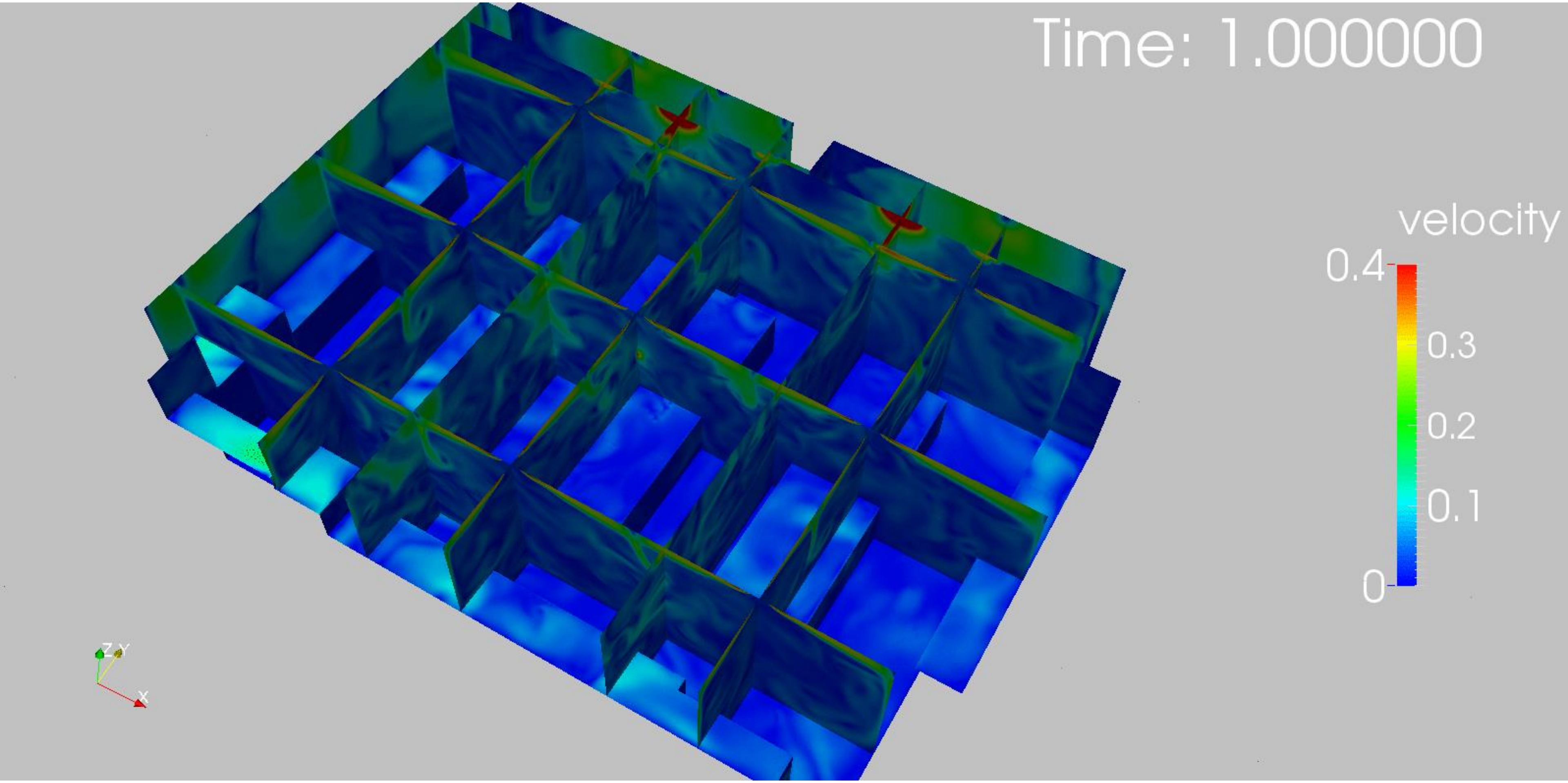
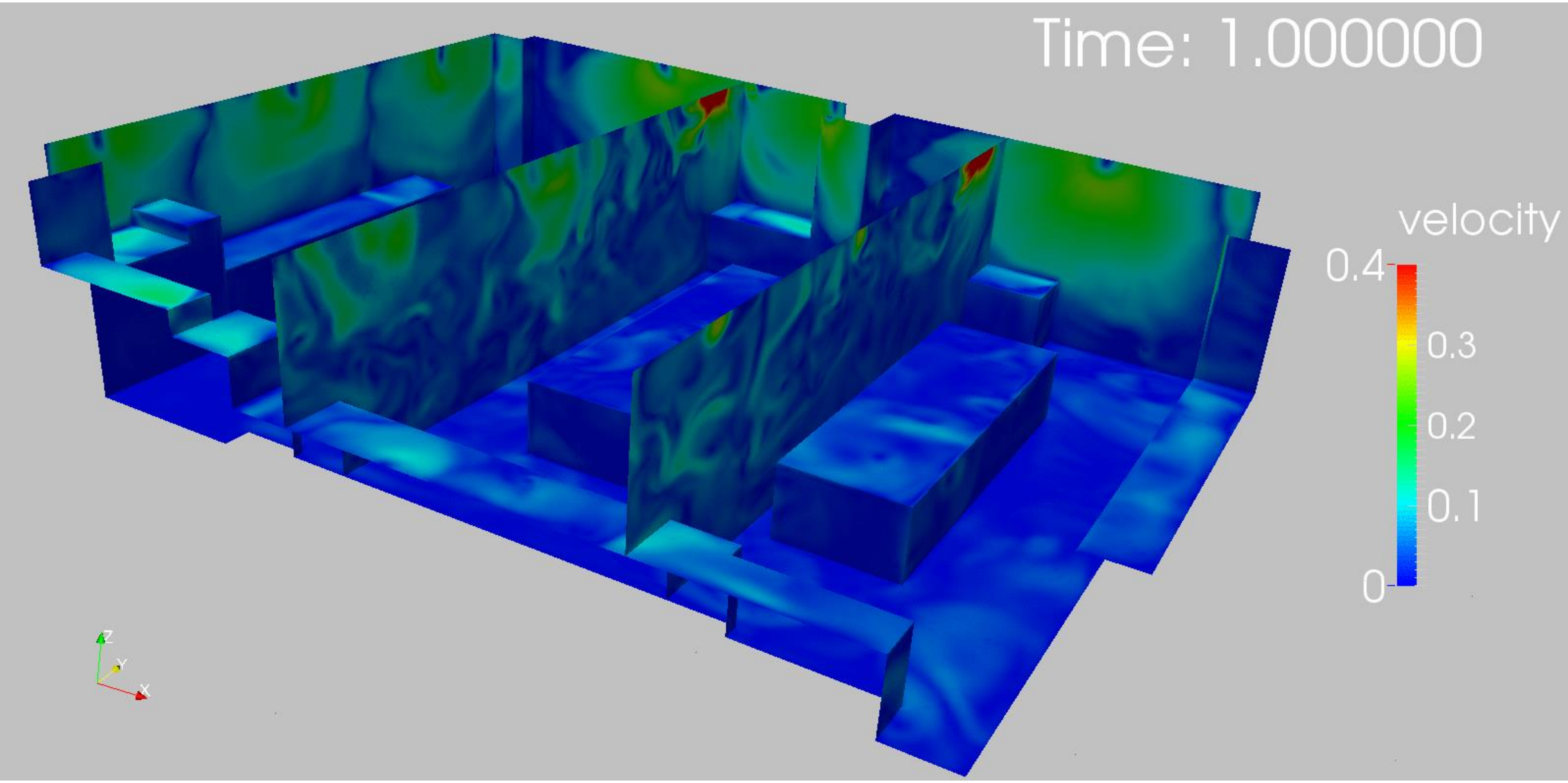


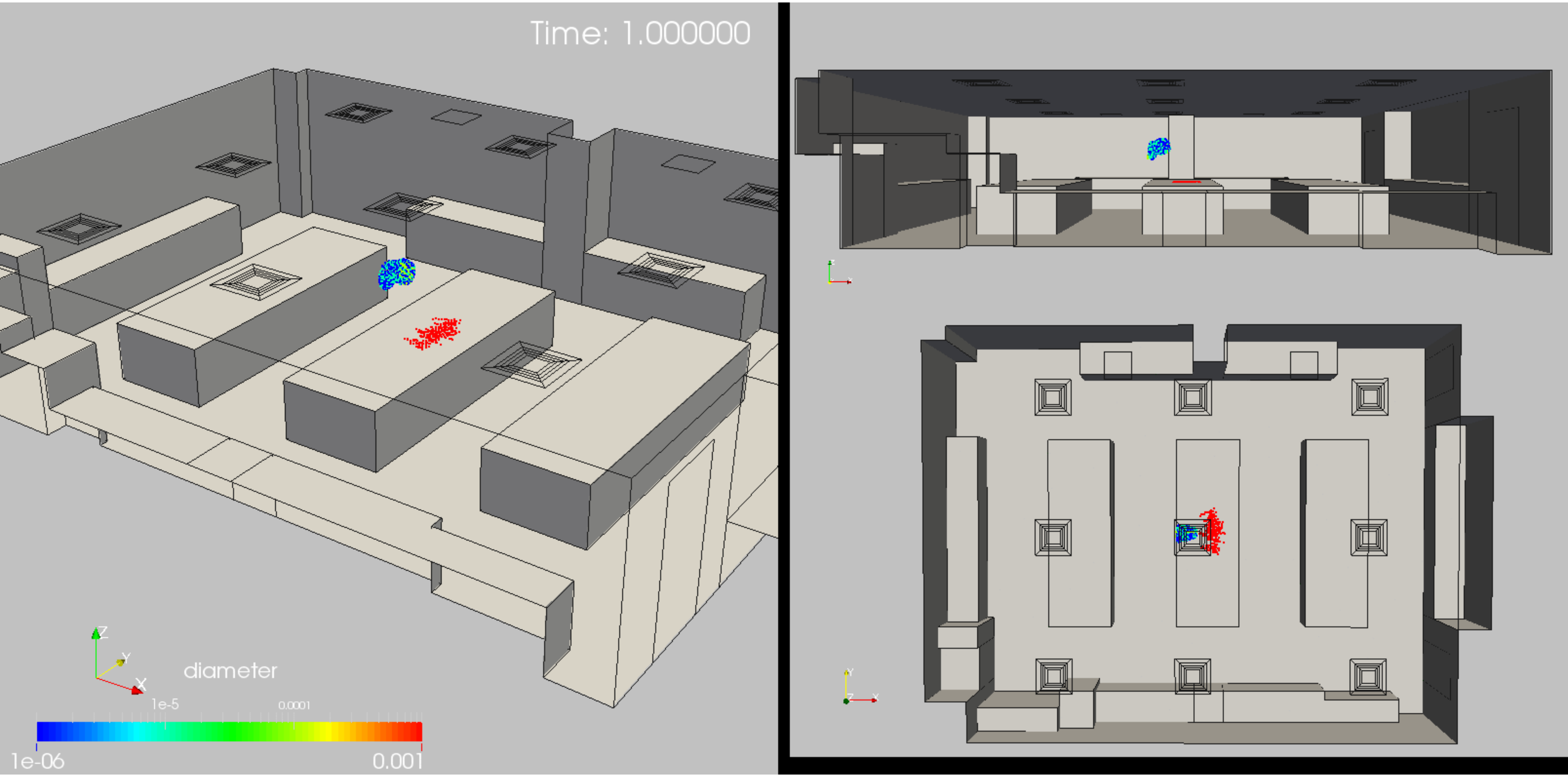
Sneezing in a Generic Classroom



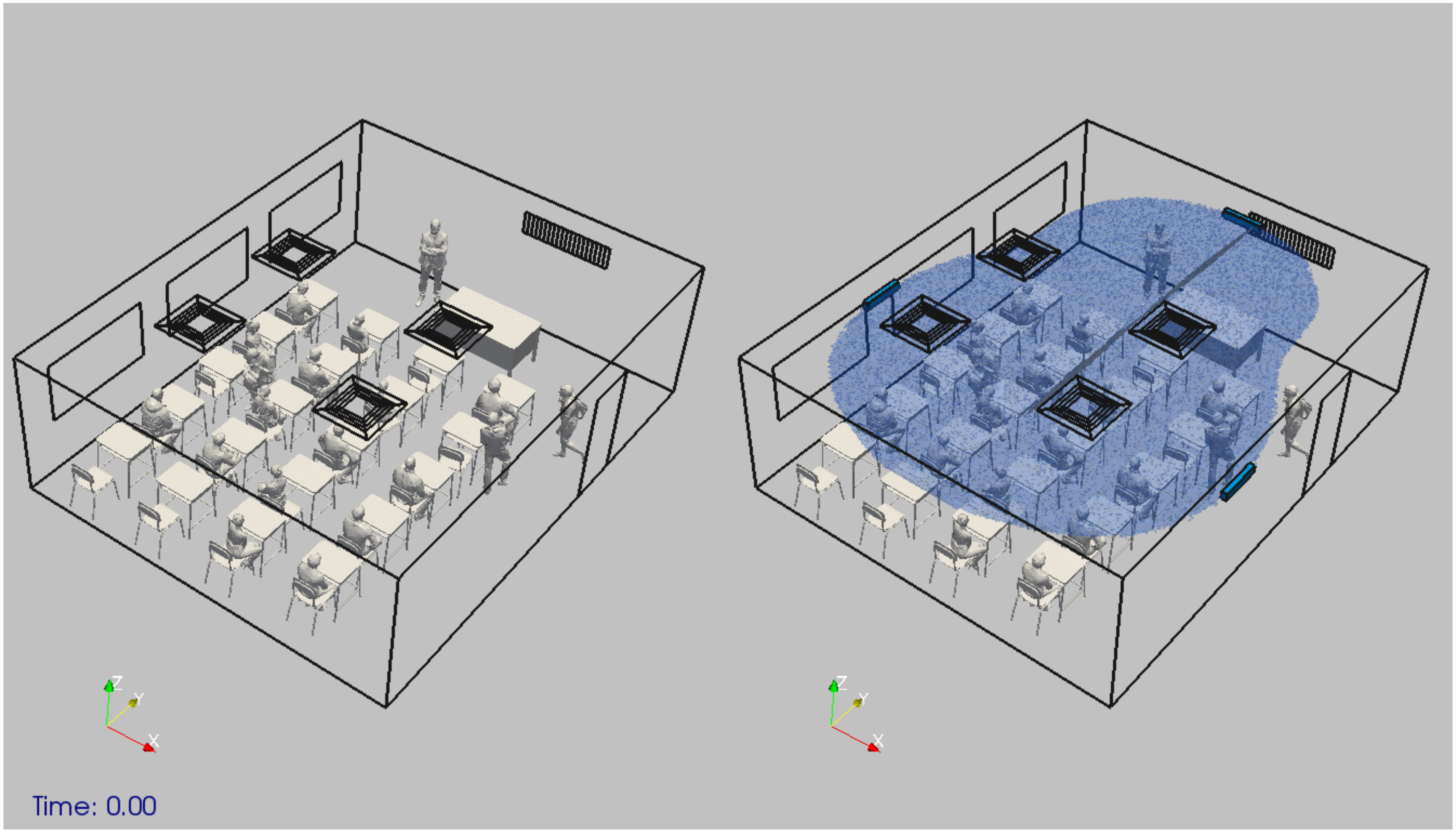
Sneezing in a Generic Classroom



Sneezing in a Generic Classroom

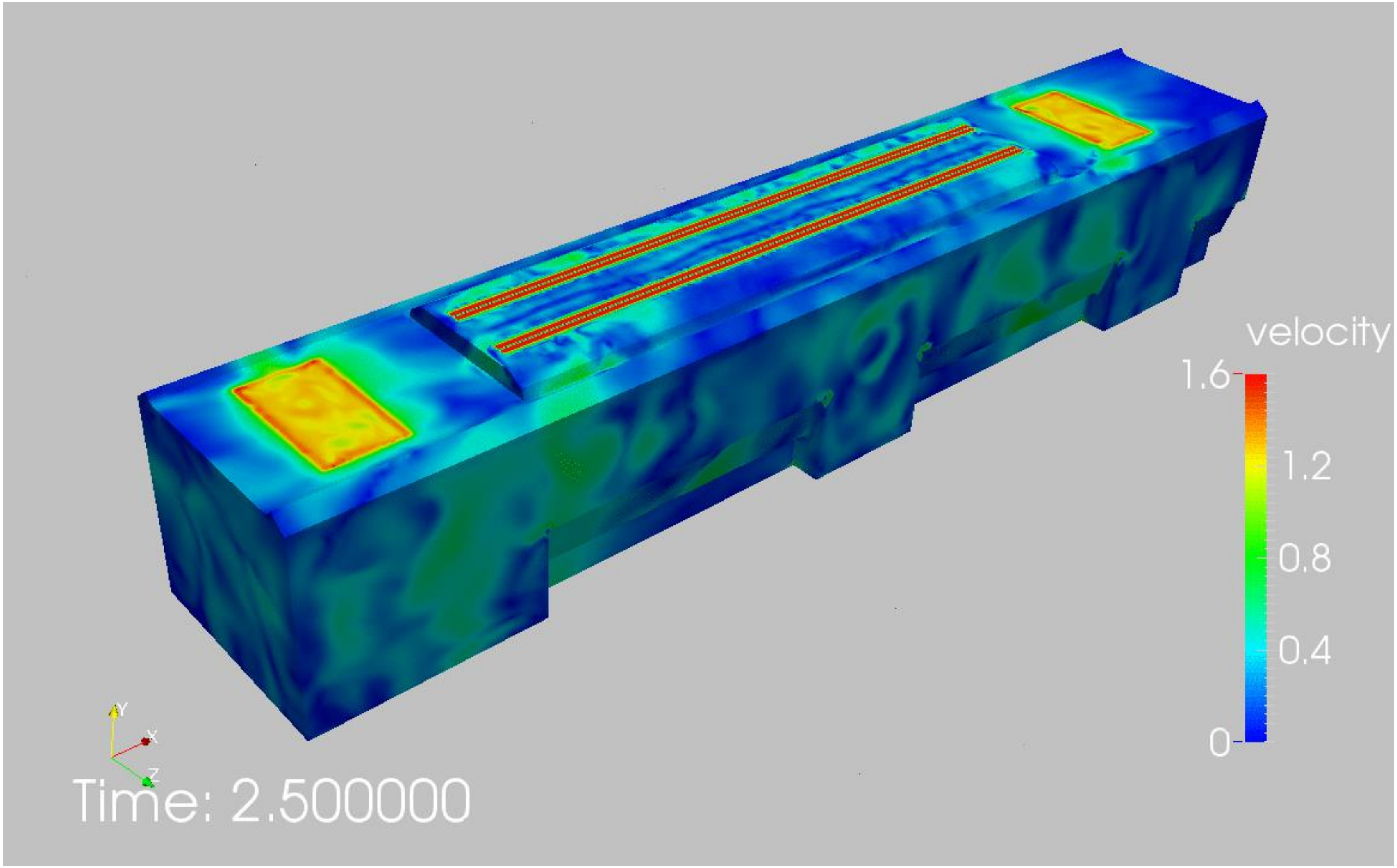


Classroom With Ceiling-UV

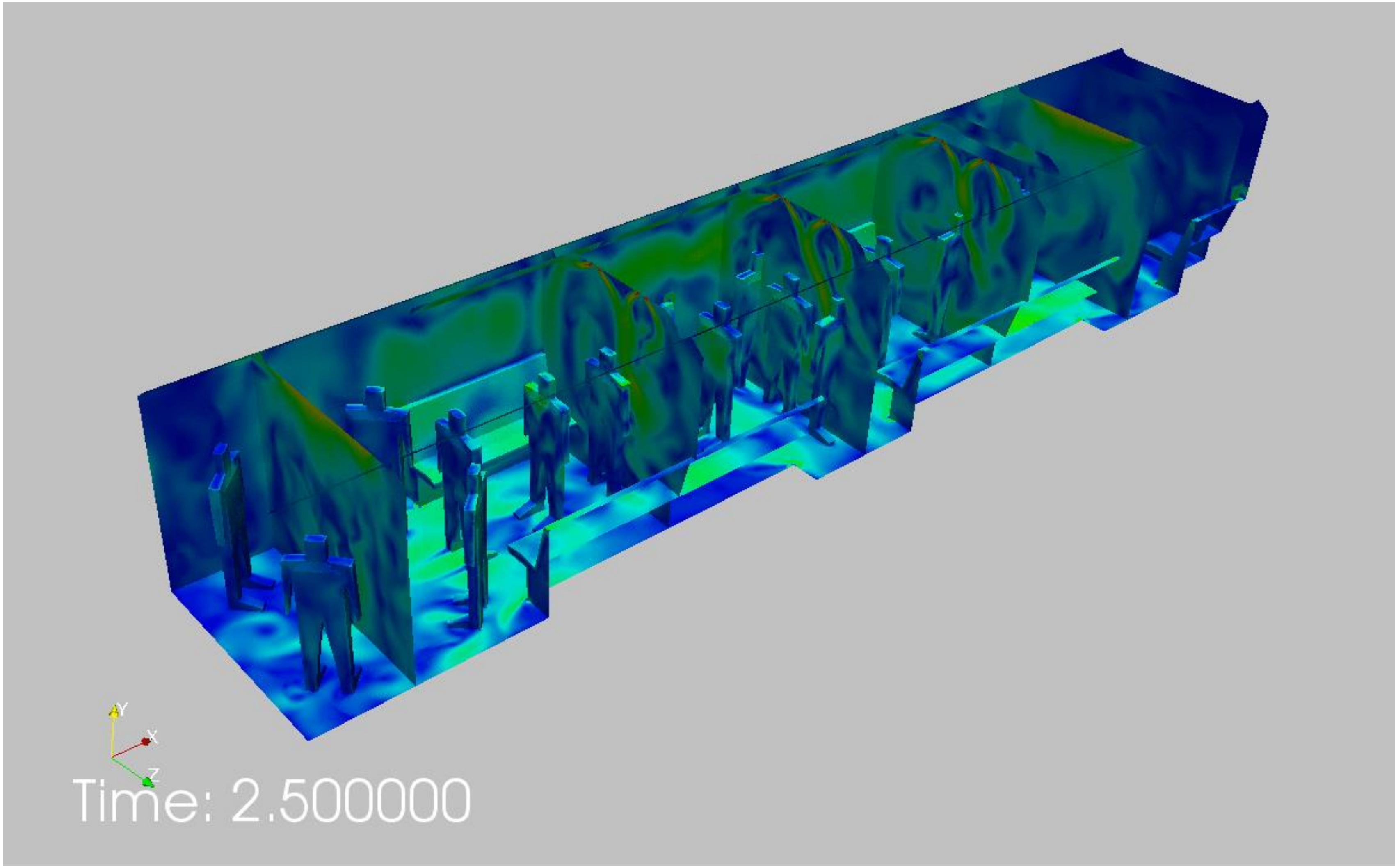


NBC Universal Today Show, September 30 (2020), see: https://www.today.com/health/ventilation-covid-19-reduce-spread-proper-airflow-t192366

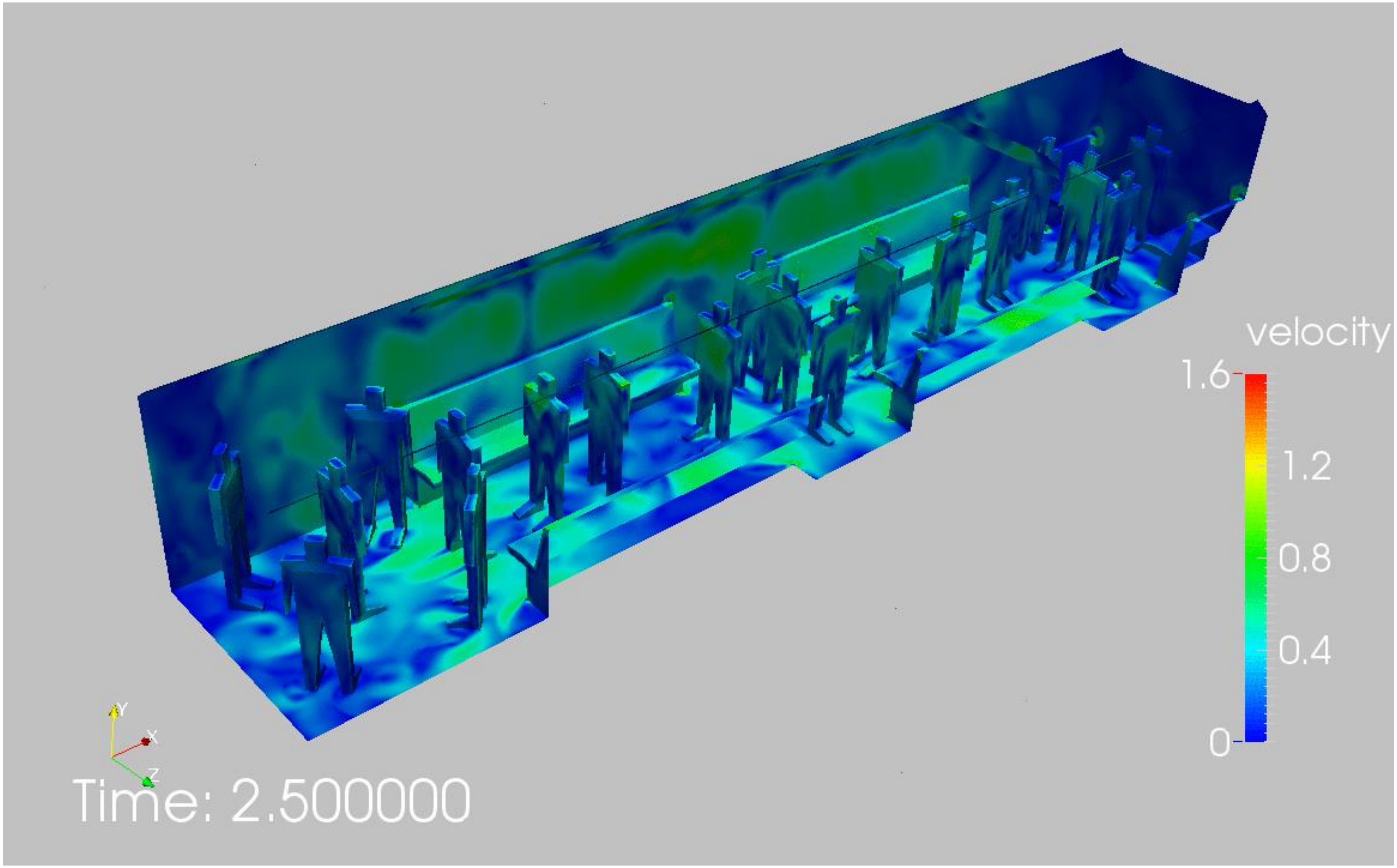
Sneezing in Subway



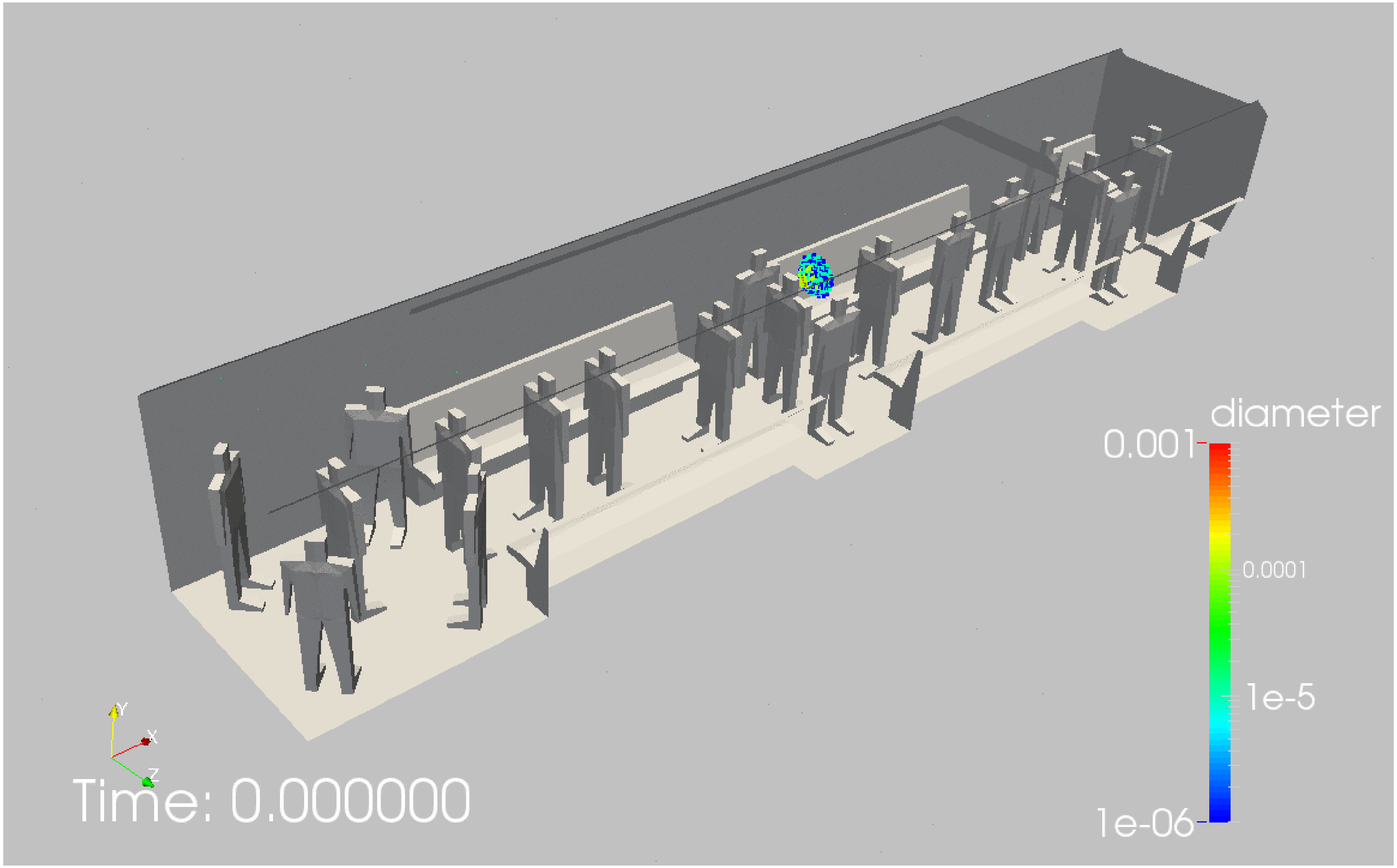
Sneezing in Subway



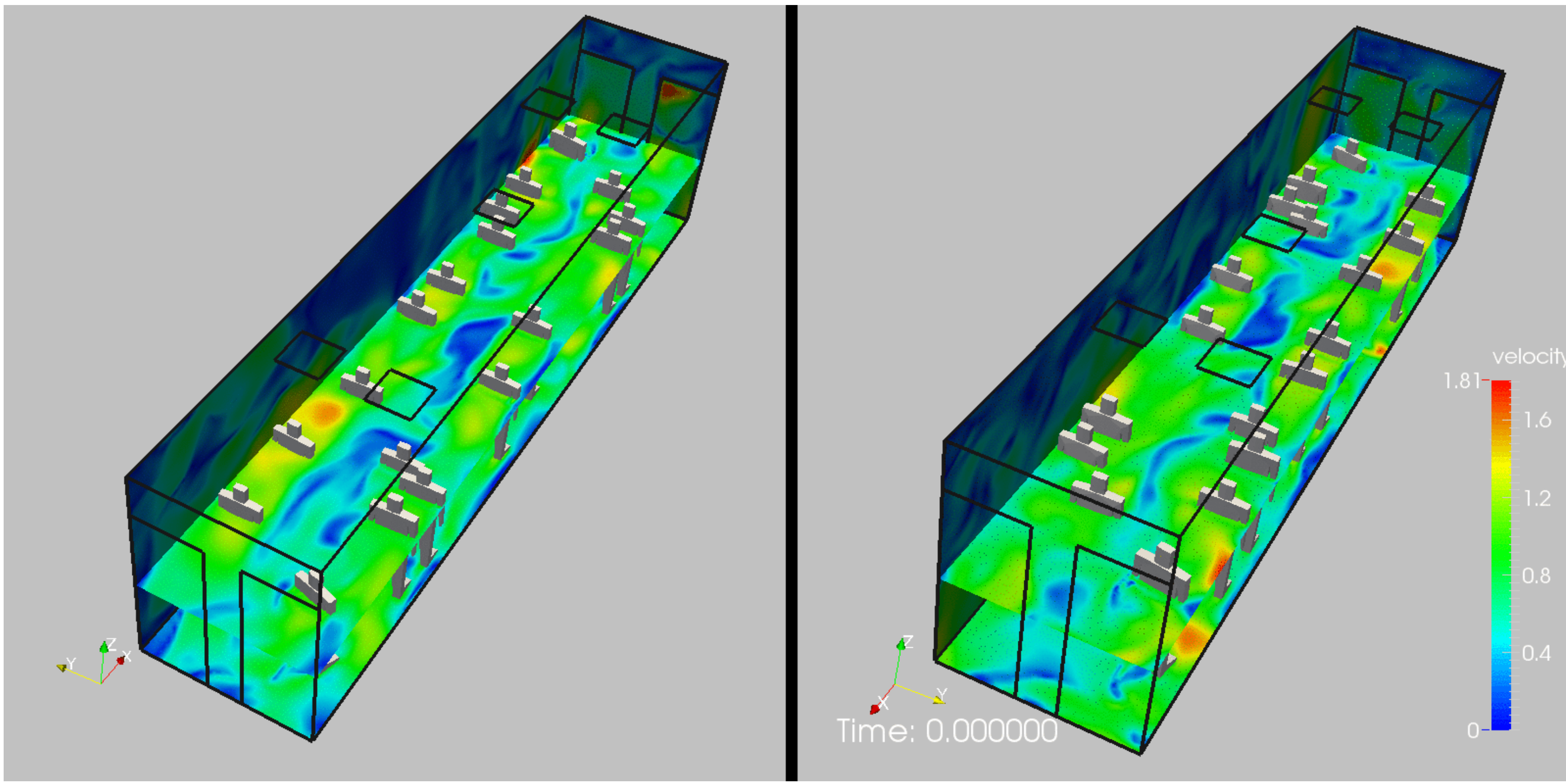
Sneezing in Subway



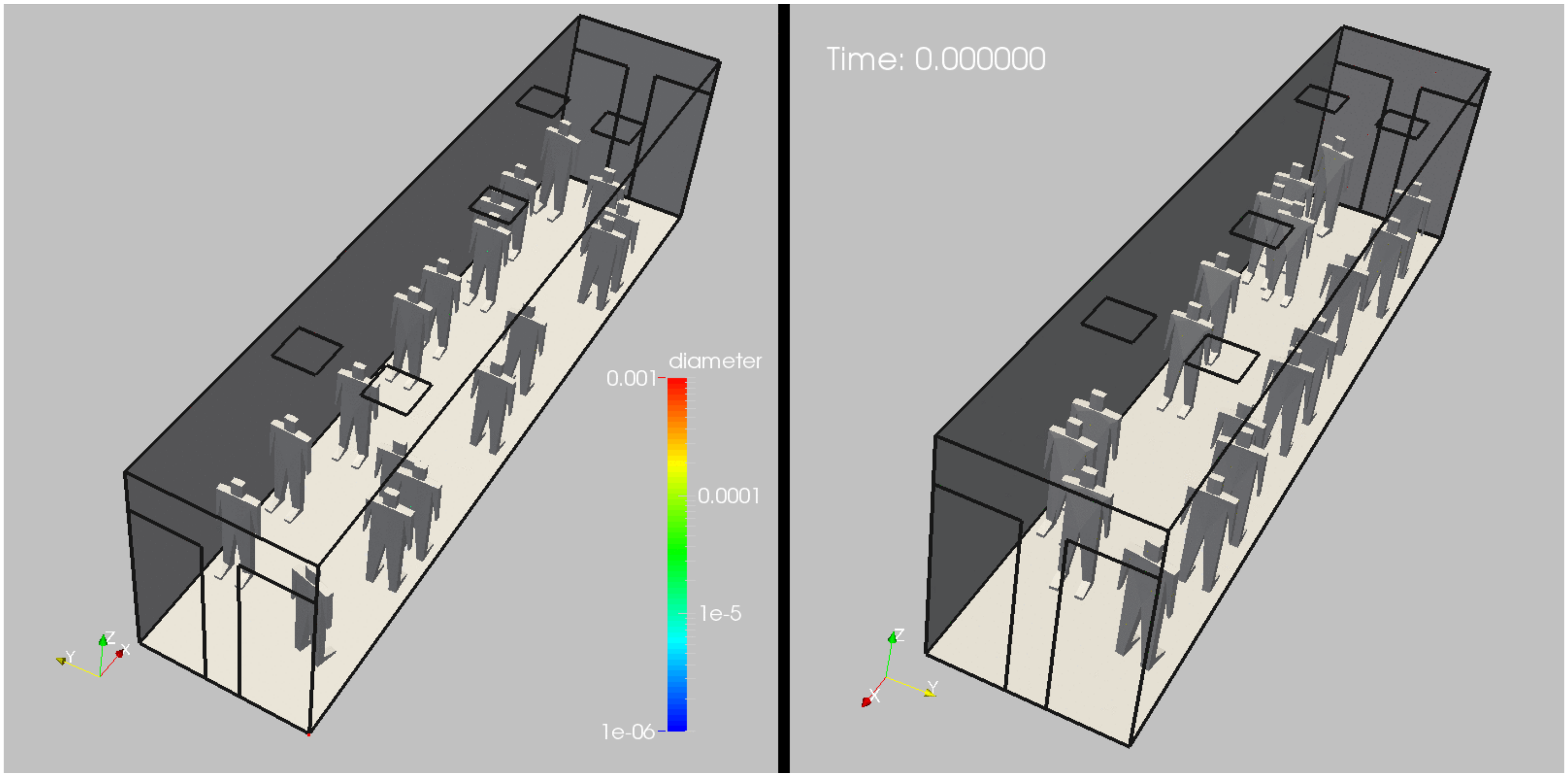
M. Gröndahl, Ch. Goldbaum and J. White - What Happens to Viral Particles on the Subway;
New York Times, August 10 (2020);
see also: https://www.nytimes.com/interactive/2020/08/10/nyregion/nyc-subway-coronavirus.html



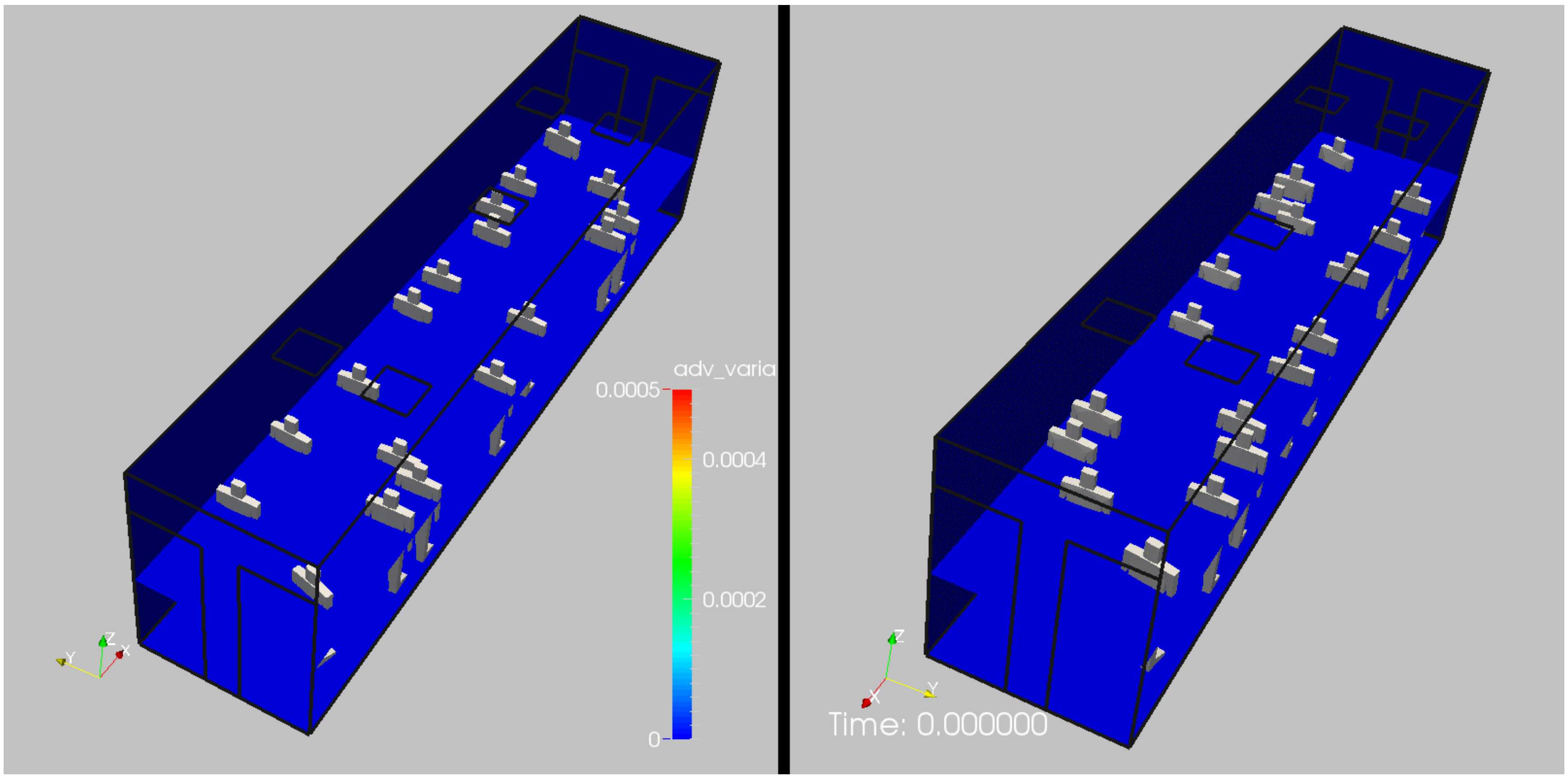
Narrow Corridor



Narrow Corridor



Narrow Corridor: Viral Load



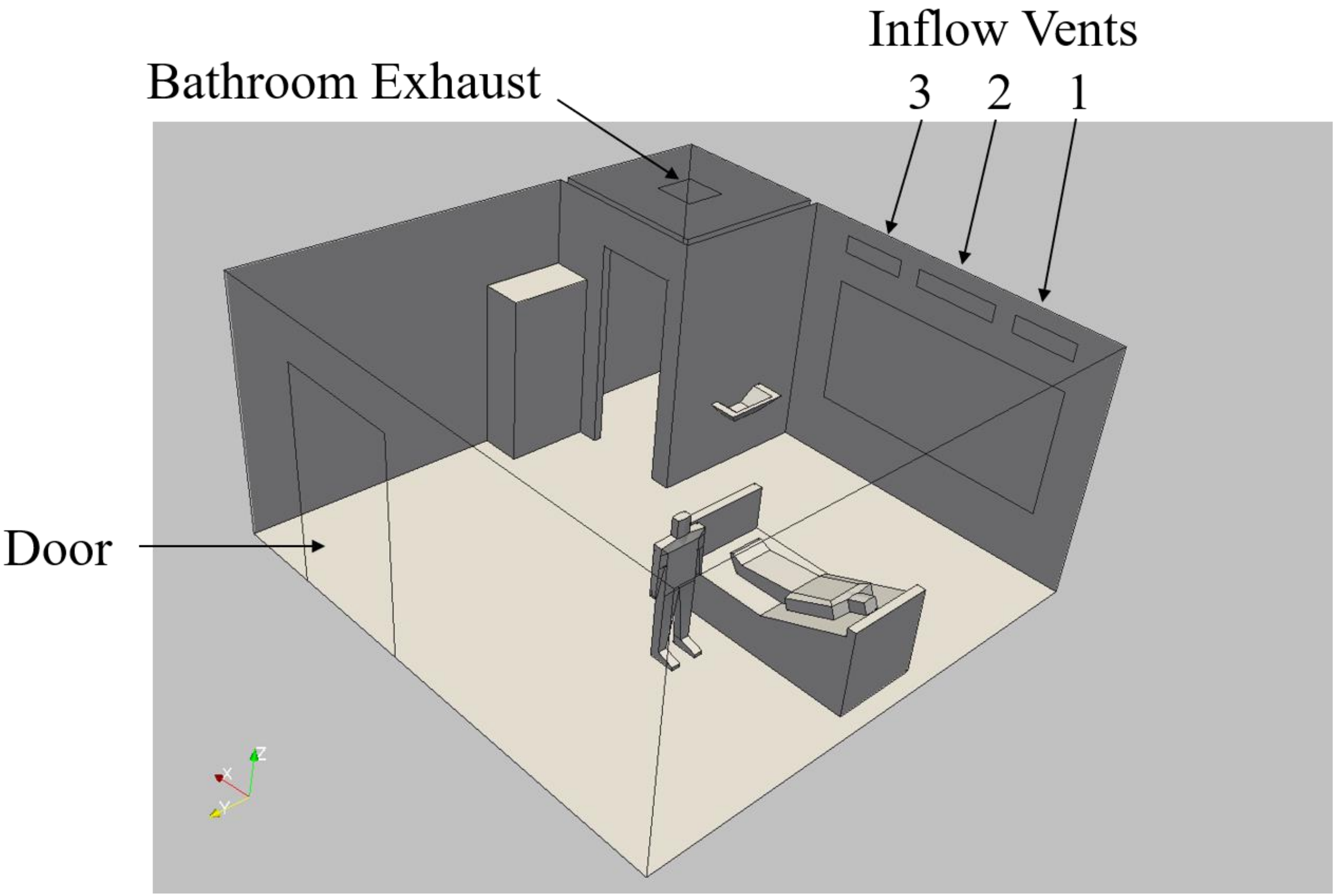
Reopening After the Crisis

Reopening After the Crisis

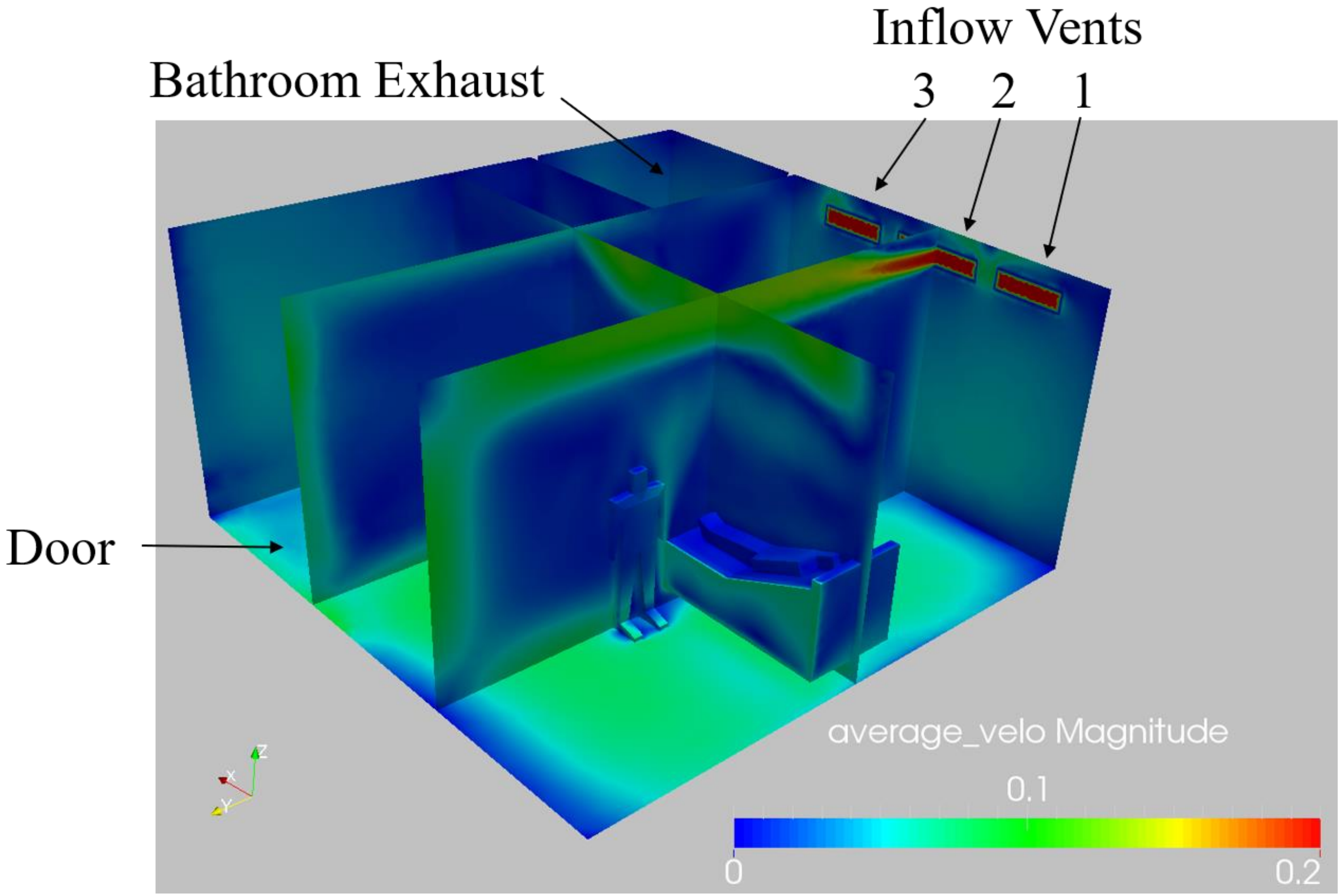
- Will Have/Need Sensors to Monitor Environment
- Basic Questions:
  - How Many ?
  - Placed Where ?

- Current Approach:
  - Run Many Scenarios Cases
  - Place Many Sensors
  - Keep (Recursively) the One Detecting the Most Cases

Hospital Room: Problem Setup



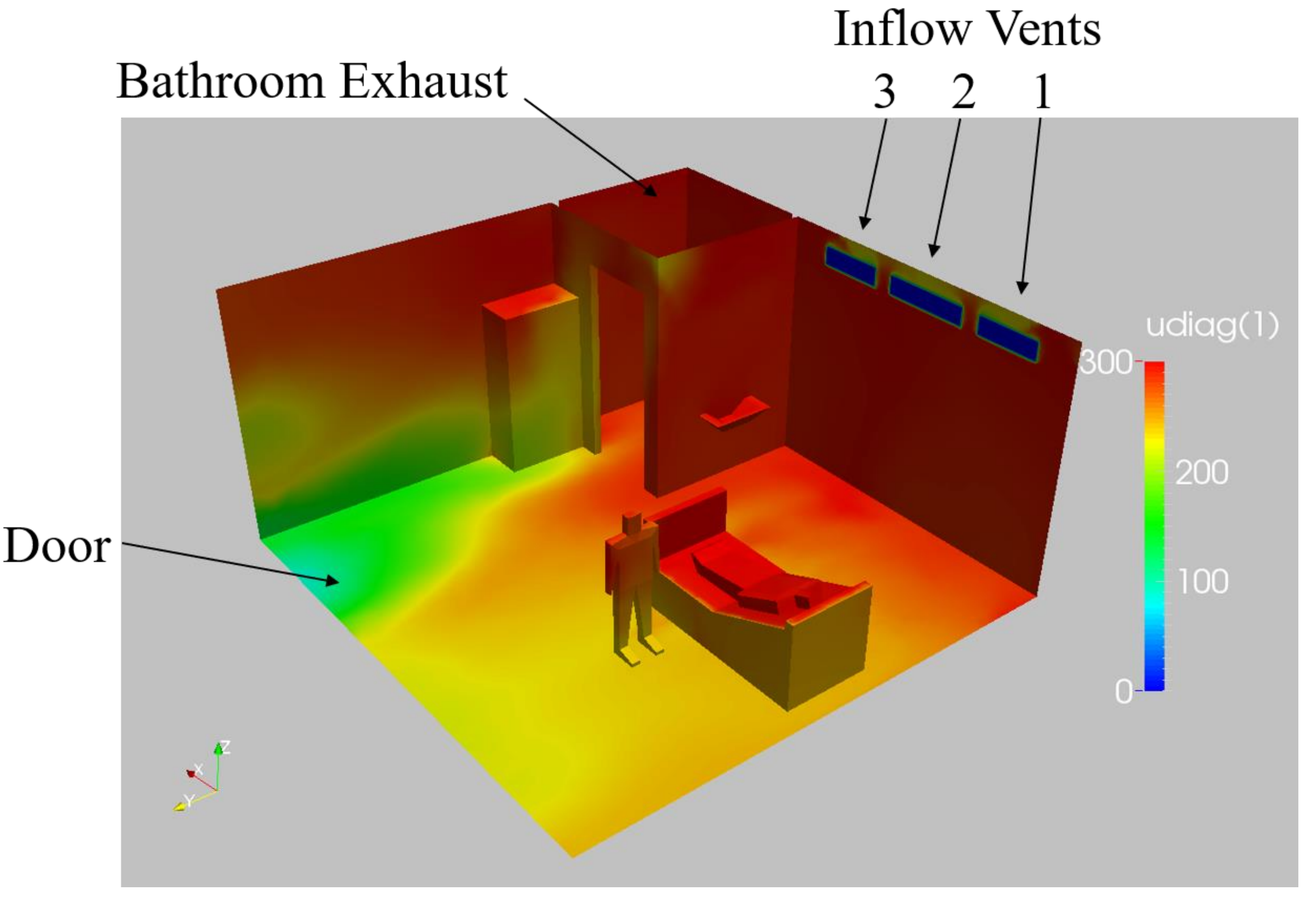
Hospital Room: Average Velocities



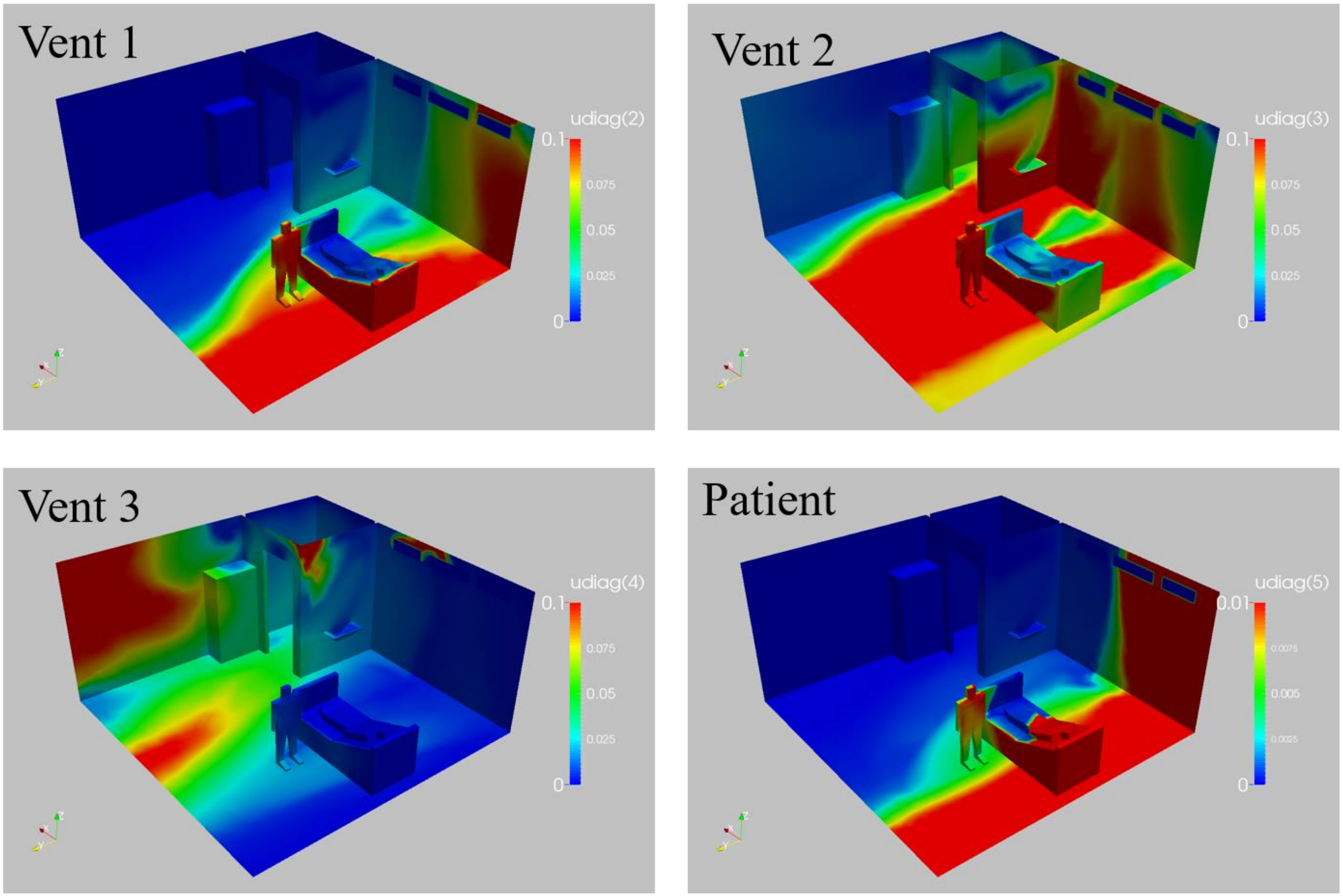
Case Study: Hospital Room

- Cases 1,2,3:
  - Contaminant/Pathogens Through Each of the 3 Different Entry Vents
  - 0-60 sec
- Case 4:
  - Virus Production from Patient
  - 0-10 sec
- Run for 300 sec (5 min) of Real Time
- Measure Contaminant/Pathogen Concentration on all Walls

Hospital Room: Age of Air



Max Values Measured During 5 Mins

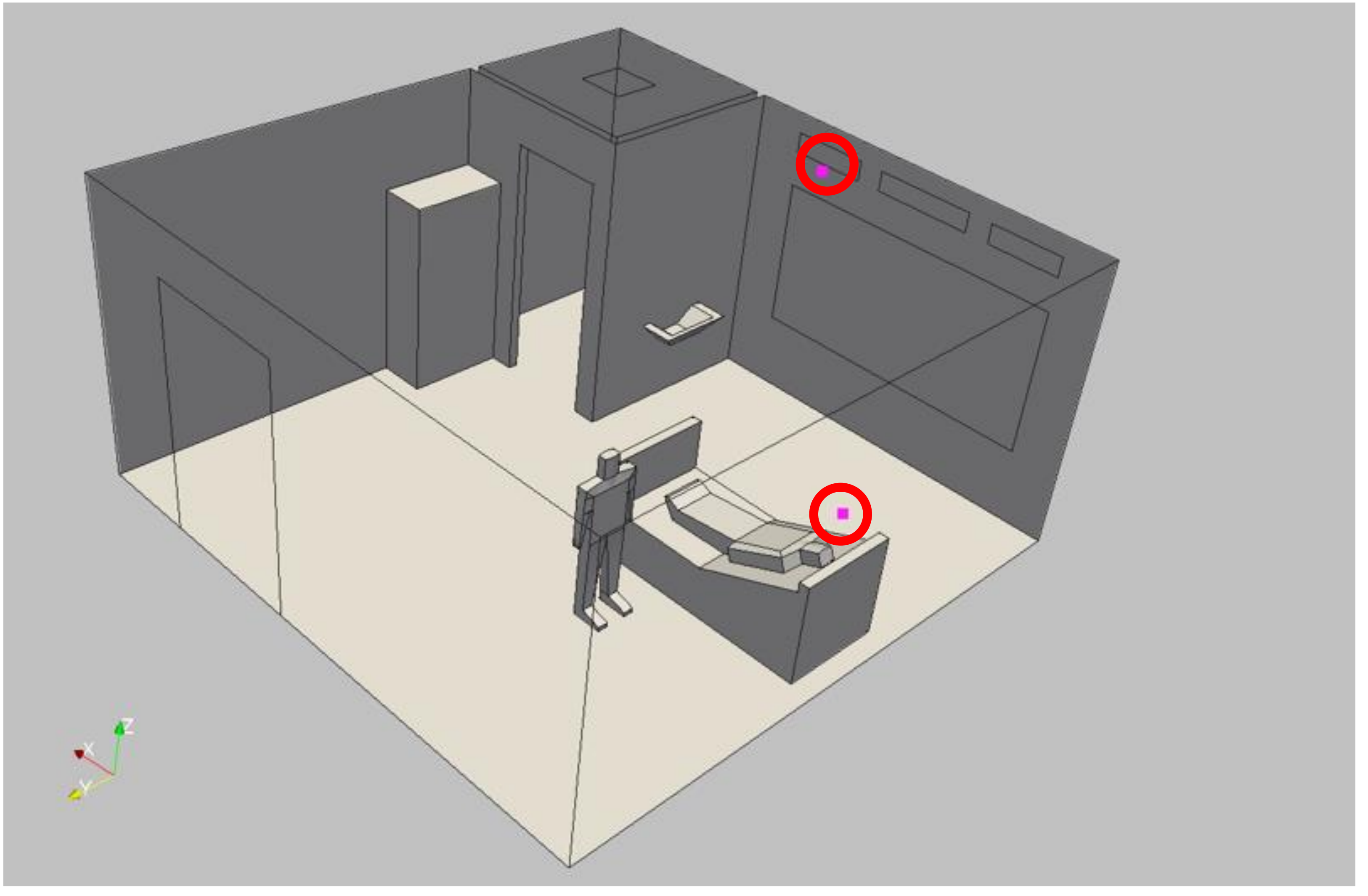


Wall Data Recorded

- Points With No Cases Measured: 4308
- Points With 1 Case (Out of 4) Measured: 3377
- Points With 2 Cases (Out of 4) Measured: 1010
- Points With 3 Cases (Out of 4) Measured: 0
- Points With 4 Cases (Out of 4) Measured: 0

- Excluded Location:
  - Above zmin (Minimum Height Requirement)
  - Not on Beds/Furniture/Patient/Attendant

Result: 2 (Optimal) Sensors



Conclusions

- Summarized:
  - Mechanical Characteristics of Virus Contaminants
  - Transmission Via Droplets and Aerosols
- Emphasis on High-Fidelity (Hi-Fi) Physics
  - PDEs
  - Appropriate Numerical Methods
- Examples from the Built Environment
  - TSA Queues, Hospital Rooms, Corridors, Trains, …
- Optimal Placement of Sensors

Outlook

- Increase Realism
  - Boundary Conditions for HVAC Systems [Entry, Mixing, …]
- Streamlining Simulation Toolbox/Workflow
- Field These Tools To Enable Smooth Post-Pandemic Transition

Thank You Very Much for Your Attention

Additional Material

Temporal Discretization: Δ-Scheme (1)

$$\mathrm{v}^{n+1} = \mathrm{v}^n + \Delta\mathrm{v}^a + \Delta\mathrm{v}^p = \mathrm{v}^* + \Delta\mathrm{v}^p$$

- Advective / Diffusive Prediction: $\mathrm{v}^n \rightarrow \mathrm{v}^*$

$$\left[\frac{1}{\Delta t} - \theta\nabla\mu\nabla\right](\mathrm{v}^* - \mathrm{v}^n) + \mathrm{v}^n \cdot \nabla\mathrm{v}^n + \nabla p^n = \nabla\mu\nabla\mathrm{v}^n$$

- Pressure Correction: $p^n \rightarrow p^{n+1}$

$$\nabla \cdot \mathrm{v}^{n+1} = 0$$

$$\frac{\mathrm{v}^{n+1} - \mathrm{v}^*}{\Delta t} + \nabla(p^{n+1} - p^n) = 0$$

$$\Rightarrow \Rightarrow \quad \nabla^2(p^{n+1} - p^n) = \frac{\nabla \cdot \mathrm{v}^*}{\Delta t}$$

Temporal Discretization: Δ-Scheme (2)

- Velocity Correction: $\mathrm{v}^* \rightarrow \mathrm{v}^{n+1}$

$$\mathrm{v}^{n+1} = \mathrm{v}^* - \Delta t \nabla (p^{n+1} - p^n)$$

<u>Remarks:</u>

- Residuals of Pressure Correction Vanish for Steady-State
  - ➔ Results Not Depend on Projection Scheme
  - ➔ Results Not Depend on Δt

Spatial Discretization: Advection (1)

- Galerkin:

$$\mathbf{r}^{\mathrm{i}} = D^{ij}\mathbf{F}_{\mathbf{ij}} = D^{ij}\left(\mathbf{f}_i + \mathbf{f}_j\right)$$

$$\mathbf{f}_i = S^{ij}{}_k F_i^k \quad , \quad S^{ij}{}_k = \frac{d_k^{ij}}{D^{ij}} \quad , \quad D^{ij} = \sqrt{d_k^{ij} d_k^{ij}}$$

$$d_k^{ij} = \frac{1}{2}\int\left(N_{,k}^i N^j - N_{,k}^j N^i\right) \quad d\Omega$$

$$\mathbf{f}_i = (S_k^{ij}\mathrm{v}_i^k)\mathbf{v}_i$$

Spatial Discretization: Advection (2)

- Need Consistent Numerical Fluxes
  - Integration Along Characteristics
  - Taylor-Galerkin/Streamline Diffusion/SUPG/GLS/…
  - Edge-Based Upwinding
- Consistent Numerical Flux

$$\mathrm{F}_{\mathrm{ij}} = \mathbf{f}_i + \mathbf{f}_j - \left|\mathrm{v}^{ij}\right|\left(\mathbf{v}_i - \mathbf{v}_j\right) \quad , \quad \mathrm{v}^{ij} = \frac{1}{2} S_k^{ij}\left(\mathrm{v}_i^k + \mathrm{v}_j^k\right)$$

- Higher Order Scheme via Limiting (e.g. MUSCL)

Spatial Discretization: Divergence (1)

- Galerkin:

$$\mathrm{r}^{\mathrm{i}} = D^{ij}\mathrm{F}_{\mathrm{ij}} = D^{ij}\left(\mathrm{f}_i + \mathrm{f}_j\right)$$

$$\mathrm{f}_i = S_k^{ij}\mathrm{v}_i^k$$

- Need Consistent Numerical Fluxes (LBB Condition)
  - Different Functional Spaces for ***v***,*p* (e.g. Mini-Element)
  - Artificial Viscosity/Stabilization
  - Edge-Based Consistent Numerical Flux

Spatial Discretization: Divergence (2)

- Consistent Numerical Flux

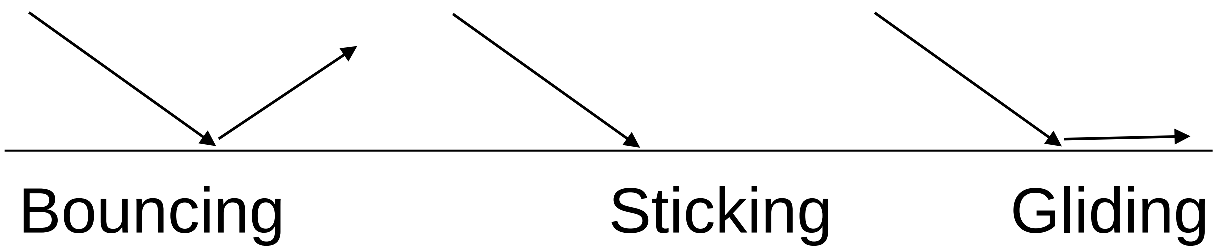
$$\mathrm{F}_{\mathrm{ij}} = \mathrm{f}_i + \mathrm{f}_j - \left|\lambda^{ij}\right|\left(p_i - p_j\right) \quad , \quad \lambda^{ij} = \frac{\Delta t^{ij}}{l^{ij}}$$

- Higher-Order (4^{th} Order Damping)

$$\mathrm{F}_{\mathbf{ij}} = \mathrm{f}_i + \mathrm{f}_j - \left|\lambda^{ij}\right|\left(p_i - p_j + \frac{\mathbf{l}^{ij}}{2}\cdot\left(\nabla p_i + \nabla p_j\right)\right)$$

Walls: Boundary Conditions

- Walls: Bouncing, Sticking, Gliding, …



- Embedded Surfaces